Teaching Introductory Linear Algebra with Open Software and Textbooks

MAA Session: Innovative and Effective Ways to Teach Linear Algebra

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Linear Algebra and Computation

An Introductory Example

$$\begin{bmatrix} -1 & 1 & 5 & -1 & -5 & 0 \\ -2 & 1 & 7 & -2 & -9 & -2 \\ 1 & 2 & 4 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 & 3 & 3 \\ 0 & -1 & -3 & 1 & 3 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -2 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Analysis of RREF

$$\begin{bmatrix} -1 & 1 & 5 & -1 & -5 & 0 \\ -2 & 1 & 7 & -2 & -9 & -2 \\ 1 & 2 & 4 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 & 3 & 3 \\ 0 & -1 & -3 & 1 & 3 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -2 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Rank 4, blue entries are 4×4 identity matrix
- Range (column space) is spanned by columns 1, 2, 4, and 6
- Kernel (null space) is spanned by 2 vectors with red entries
- As a coefficient matrix, solutions are pre-images
  Solution iff last column row-reduces with zero entry row 5
- A left null space vector produces zero row via linear combo

RREF in $\mathbb{C}^3$

Suppose A is a 3×3 matrix (with no zero columns), then its reduced row-echelon form looks like:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & \times \\ 0 & 1 & \times \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & \times & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & \times & \times \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The **geometric** intuition of 3 dimensions is useful, but there is not much **algebraic** variety or generality here.

Conclusions

Session Description:
"(5) comparing and contrasting visual (geometric) and more abstract (algebraic) explanations of specific ideas"

- Analysis of "large" matrices are crucial for an algebraic approach
- We do not want students computing the RREF of a 5×6 matrix by hand, so computational tools are an important part of an introductory course
- Computations should be **exact**, so for an *introductory* course, the field of rational numbers is perfect
(not the reals, not the complexes)
- Sage (open source Mathematica, Maple, Matlab, Magma) fits the bill with very thorough support over the rationals

Open Software and Textbooks

Extreme Example: Eigenvalues of a Matrix from Sage

```
sage: matrix(QQ, [[10, -12, -11, -13],
                  [12, -13,  -9, -12],
                  [-6,   9,  13,  14],
                  [ 2,  -5, -11, -11]])

sage: A.fcp()
(x + 1) * (x^3 - 2)

sage: A.eigenvalues()
[-1, 1.259921049894873?,
-0.6299605249474365? - 1.091123635971722?*I,
-0.6299605249474365? + 1.091123635971722?*I]

sage: N(A.eigenvalues()[1]^3, digits=45)
2.00000000000000000000000000000000000000000000
```

Sage Cells in Open Source Textbook

https://linear.ups.edu/fcla/section-EE.html

≡ Contents | Index | < Prev | ^ Up | Next >

Authored in PreTeXt

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discover these eigenvectors ourselves.

Example SEE: Some eigenvalues and eigenvectors.

Sage EE Eigenvalues and Eigenvectors

Sage EE Eigenvalues and Eigenvectors. Sage can compute eigenvalues and eigenvectors of matrices. We will see shortly that there are subtleties involved with using these routines, but here is a quick example to begin with. These two commands should be enough to get you started with most of the early examples in this section. See the end of the section for more comprehensive advice.

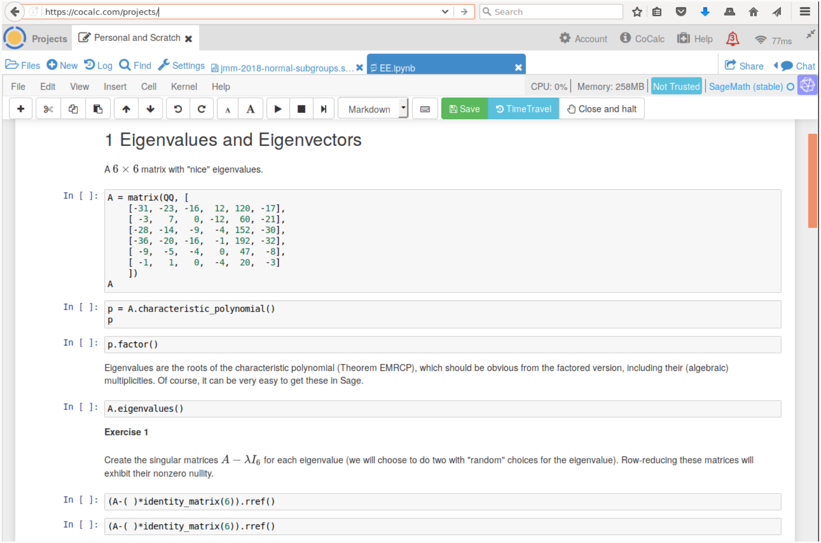
For a square matrix, the methods `.eigenvalues()` and `.eigenvectors_right()` will produce what you expect, though the format of the eigenvector output requires some explanation. Here is Example SEE from the start of this chapter.

```
A = matrix(QQ, [[ 204,   98, -26, -10],
                [-280, -134,  36,  14],
                [ 716,  348, -90, -36],
                [-472, -232,  60,  28]])
A.eigenvalues()
```

Evaluate Sage Code

```
[4, 0, 2, 2]
```

In-Class Demonstrations



1 Eigenvalues and Eigenvectors

A 6×6 matrix with "nice" eigenvalues.

```
In [ ]: A = matrix(QQ, [
            [-31, -23, -16,  12, 120, -17],
            [ -3,   7,   0, -12,  60, -21],
            [-28, -14,  -9,  -4, 152, -30],
            [-36, -20, -16,  -1, 192, -32],
            [ -9,  -5,  -4,   0,  47,  -8],
            [ -1,   1,   0,  -4,  20,  -3]
            ])
        A
```

```
In [ ]: p = A.characteristic_polynomial()
        p
```

```
In [ ]: p.factor()
```

Eigenvalues are the roots of the characteristic polynomial (Theorem EMRCP), which should be obvious from the factored version, including their (algebraic) multiplicities. Of course, it can be very easy to get these in Sage.

```
In [ ]: A.eigenvalues()
```

Exercise 1

Create the singular matrices $A - \lambda I_6$ for each eigenvalue (we will choose to do two with "random" choices for the eigenvalue). Row-reducing these matrices will exhibit their nonzero nullity.

```
In [ ]: (A-( )*identity_matrix(6)).rref()
```

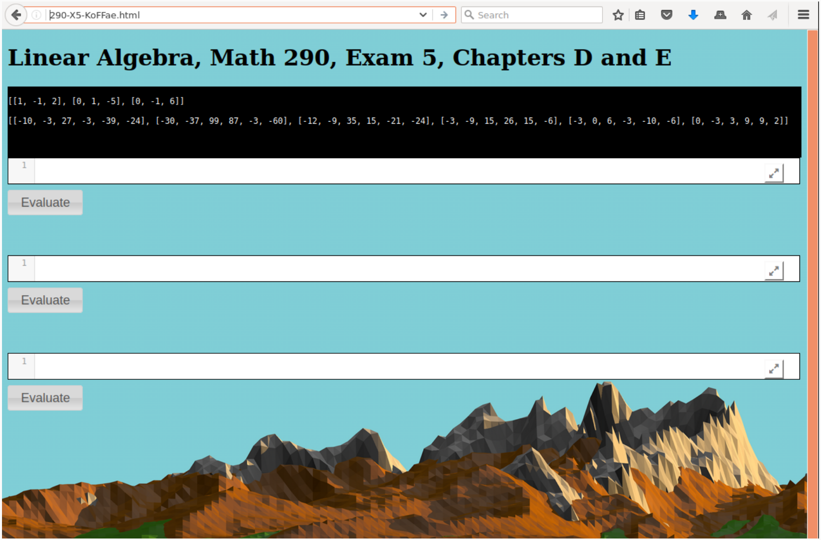
```
In [ ]: (A-( )*identity_matrix(6)).rref()
```

Examinations

- Students use the full range of powerful Sage commands to study the subject, for example `A.column_space()`.
- For examinations solutions are typically limited to:
    - Vector and matrix operations (products, transpose, etc.)
    - Reduced row-echelon form
    - Determinant
    - Factored characteristic polynomial
    - Eigen-stuff
- During exams: students' laptops,
  plus provided web page with matrix inputs and Sage cells

PS: Sage is useful for constructing examinations,
especially "random" matrices with "nice" properties
(integer RREF, integer eigenvalues, determinant 1, etc.)

Sample Examination Calculator



Resources

Freely available, with open licenses:

Textbook	`linear.pugetsound.edu`
Classroom Demos	`github.com/rbeezer/sla`
Sage	`sagemath.org`
CoCalc	`cocalc.com`
Slides	`buzzard.ups.edu/talks.html`

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