A Modern Online Linear Algebra Textbook

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18th Conference
of the
International Linear Algebra Society
Providence, Rhode Island
June 7, 2013

Introduction and Outline

Two parts:

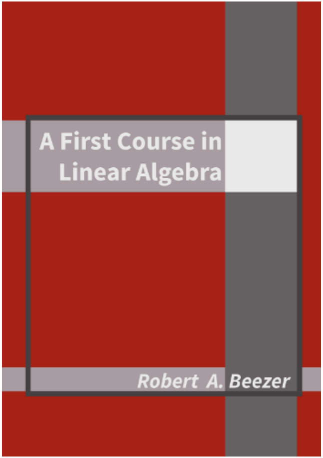
- Thoughts on organizing an introductory course
- Modern approach to textbook design and distribution

- Follow along in the third half:

  `http://linear.ups.edu`, left sidebar: "Online"

- Support: NSF TUES Grant, UTMOST project, `utmost.aimath.org`
- Support: Shuttleworth Foundation Flash Grant

A First Course in Linear Algebra



- Initiated 2003; Version 1.0 2006
- Always free online
- GNU Free Documentation License
- Sophomore course
- Emphasis on proof techniques

Chapter: System of Equations

- Best motivation for students coming out of calculus
- Hint: reduced row-echelon form is a column-by-column algorithm
- Natural place to introduce null spaces and nonsingular matrices
- Cycle back and rephrase in the language of the linear transformation

$$T : \mathbb{C}^n \to \mathbb{C}^m \quad T(\mathbf{x}) = A\mathbf{x}$$

Chapter: Vectors

- A vector space has addition and scalar multiplication
- So a linear combination is the most natural construction
- Spanning sets and linear independence follow

- Other consequences:
  - Product of a matrix A and a vector $\mathbf{x}$ is the linear combination of the columns of A with scalars from the entries of $\mathbf{x}$
  - Matrix multiplication:

$$AB = A[B_1|B_2|\ldots|B_p] = [AB_1|AB_2|\ldots|AB_p]$$

  - The entry-by-entry formula for a matrix product,

$$\sum_j a_{ij}b_{jk}$$

    is now a theorem, derived from linear combinations

Chapter: Matrices

- Matrix operations, multiplication, inverses
- Various subspaces just as **sets**
  Treat as **vector spaces** later
  (spans, column space, row space, null space, left null space)
- When to consider orthogonality?
  - Vectors: orthogonal pairs, orthogonal sets, Gram-Schmidt
  - Matrices: adjoint, Hermitian (self-adjoint), unitary

- Extended Echelon Form of $m \times n$ matrix A (perhaps rectangular)

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \begin{bmatrix} C & K \\ 0 & L \end{bmatrix}$$

- Matrix on right (J) records row-operations, canonically
- L has rows which record “zero-ing” of rows of A

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  - The null space of A is the null space of C; dimension $n - r$
  - The row space of A is the row space of C; dimension r
  - The column space of A is the null space of L; dimension r
  - The left null space of A is the row space of L; dimension $m - r$

Chapters: Determinants, Eigenvalues

- Eigenvalues are necessarily complex numbers, even if we use $\mathbb{R}^n$
- $A\mathbf{x} = \lambda\mathbf{x}$ then introduces vectors with complex entries
- So consistently work over $\mathbb{C}^n$ rather than $\mathbb{R}^n$
  - No penalty to do so
  - Do not need to use complex numbers for examples
  - Better inner product (using complex conjugation)
  - Some theorems easier (algebraically closed field)

Chapter: Vector Spaces

- Have many examples of subspaces in $\mathbb{C}^n$
- Can now formulate more axiomatic treatment
- Key theorem for properties of dimension

If a set of t vectors spans the vector space V, then any set of $t+1$ or more vectors is linearly dependent.

Chapter: Linear Transformations

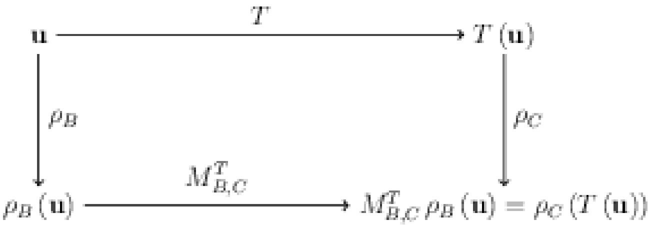
- Heavy use of pre-images (a set)
- Parallels early theorems about solutions to systems of equations
- Inverse of a linear transformation
  - Surjective: pre-images are all non-empty
  - Injective: pre-images have at most one element
  - Bijective: each pre-image is a singleton, so use this to establish existence of the inverse linear transformation constructively
  - Then exercises construct inverse linear transformations from pre-images of a basis of the codomain

Chapter: Representations

Vector representation is an invertible linear transformation

- Vector space V of dimension n with basis $B = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$
- $\rho_B : V \to \mathbb{C}^n$
- $\rho_B(\mathbf{v}) = \rho_B\left(\sum_{i=1}^{n} a_i\mathbf{w}_i\right) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix}$
- Having ρ^{-1} is convenient (just a linear combination)

- Fundamental Theorem of Matrix Representation
  - Matrix representation: $M^{T}_{B,C}$
    (B, C bases of domain and codomain, respectively)
  - Then: $\rho_C(T(\mathbf{u})) = M^{T}_{B,C}(\rho_B(\mathbf{u}))$
  - Or: $T(\mathbf{u}) = \rho_C^{-1}\left(M^{T}_{B,C}(\rho_B(\mathbf{u}))\right)$



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Worldwide Audience

Most recent visitors to book content, last weekend
(09:51:55 29 May to 10:03:33 1 Jun, 2013)

A First Course in Linear Algebra, Online

Version 3.00, December 2012

- Source converted to XML
- Web version optimized for online viewing
- Standard XHTML, CSS, JavaScript ("platform-independent")
- Heavy cross-referencing
- Increased navigational aids
- Knowls: theorems, proofs, examples, exercises
- Sage cells: embedded, editable, computational examples

TEXTBOOK DEMO

`linear.ups.edu`, left sidebar: “Online”

XML Source

Section NM, Nonsingular Matrices
Theorem NMRRI, Nonsingular Matrices Row-reduce to the Identity Matrix

Save | Save As | Close | Undo | Redo

```xml
<theorem acro="NMRRI">
<title>Nonsingular Matrices Row Reduce to the Identity matrix</title>
<statement>
<p>Suppose that $A$ is a square matrix and $B$ is a row-equivalent matrix in reduced row-echelon form.
Then $A$ is nonsingular if and only if $B$ is the identity matrix.</p>
</statement>

<proof>

<p><implyreverse />  Suppose $B$ is the identity matrix.  When the augmented matrix $\augmented{A}
{\zerovector}$ is row-reduced, the result is $\augmented{B}{\zerovector}=\augmented{I_n}{\zerovector}$.
The number of nonzero rows is equal to the number of variables in the linear system of equations
$\linearsystem{A}{\zerovector}$, so $n=r$ and <acroref type="theorem" acro="FVCS" /> gives $n-r=0$ free
variables.  Thus, the homogeneous system $\homosystem{A}$ has just one solution, which must be the
trivial solution.  This is exactly the definition of a nonsingular matrix (<acroref type="definition"
acro="NM" />).</p>

<p><implyforward />  If $A$ is nonsingular, then the homogeneous system $\linearsystem{A}{\zerovector}$
has a unique solution, and has no free variables in the description of the solution set.  The
homogeneous system is consistent (<acroref type="theorem" acro="HSC" />) so <acroref type="theorem"
acro="FVCS" /> applies and tells us there are $n-r$ free variables.  Thus, $n-r=0$, and so $n=r$.  So
$B$ has $n$ pivot columns among its total of $n$ columns.  This is enough to force $B$ to be the
$n\times n$ identity matrix $I_n$ (see <acroref type="exercise" acro="NM.T12" />).</p>

</proof>
</theorem>
```

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- But: our students expect a second look

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- A usable system to author textbooks in XML (this summer)

FCLA: http://linear.pugetsound.edu

Web: http://buzzard.pugetsound.edu/talks.html

Blog: http://beezers.org/blog/bb