An Introduction to Algebraic Graph Theory

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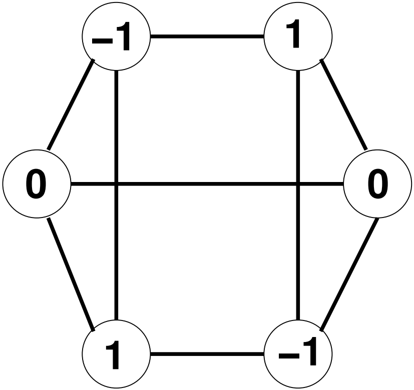
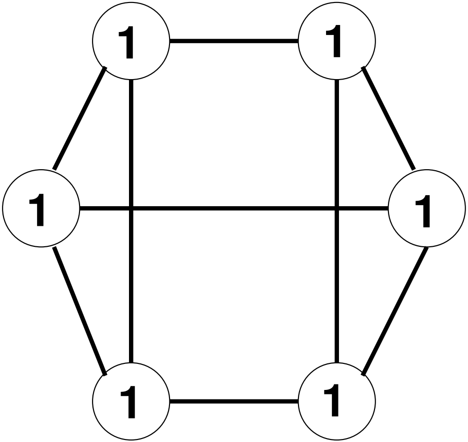
Labeling Puzzles

- Assign a single real number value to each circle.
- For each circle, sum the values of adjacent circles.
- Goal:
  Sum at each circle should be a common multiple of the value at the circle.



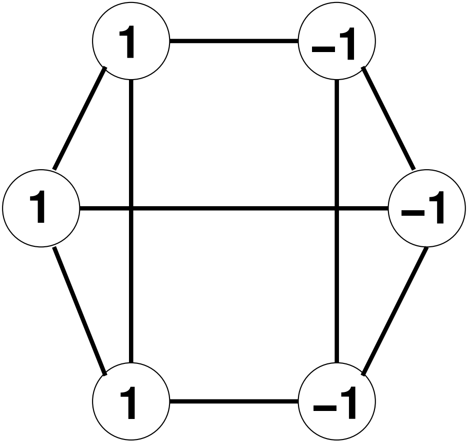
Common multiple: -2

Example Solutions



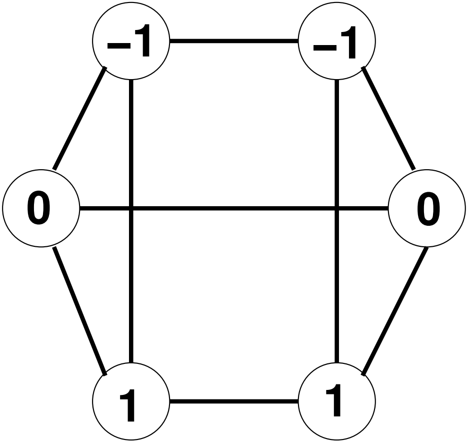
Common multiple: 3

Example Solutions



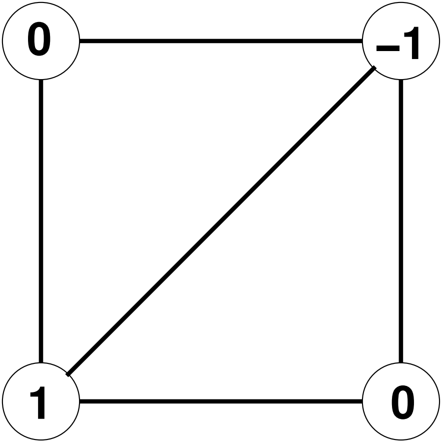
Common multiple: 1

Example Solutions



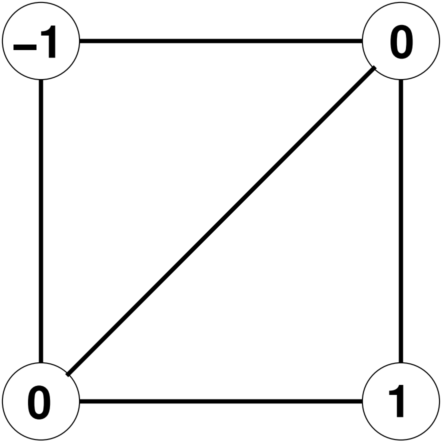
Common multiple: 0

Example Solutions



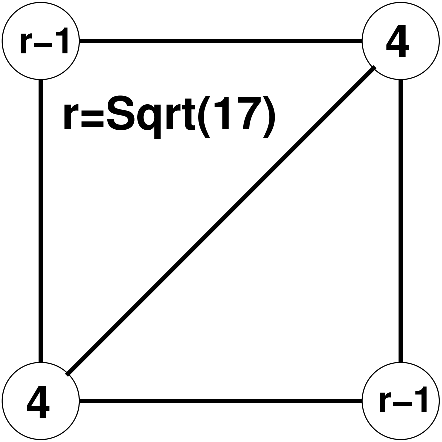
Common multiple: -1

Example Solutions



Common multiple: 0

Example Solutions

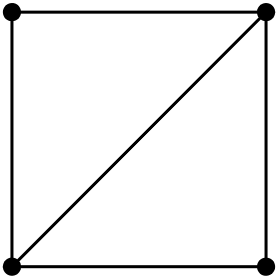


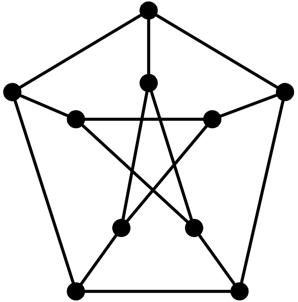
Common multiple: $\frac{1}{2}(r+1) = \frac{1}{2}\left(\sqrt{17}+1\right)$

Graphs

A graph is a collection of vertices (nodes, dots) where some pairs are joined by edges (arcs, lines).

The geometry of the vertex placement, or the contours of the edges are irrelevant. The relationships between vertices *are* important.





Adjacency Matrix

Given a graph, build a matrix of zeros and ones as follows:

Label rows and columns with vertices, in the same order.

Put a 1 in an entry if the corresponding vertices are connected by an edge.

Otherwise put a 0 in the entry.

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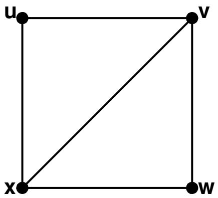

	u	v	w	x
u	0	1	0	1
v	1	0	1	1
w	0	1	0	1
x	1	1	1	0

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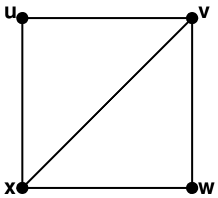

	u	v	w	x
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v	1	0	1	1
w	0	1	0	1
x	1	1	1	0

- Always a symmetric matrix with zero diagonal.
- Useful for computer representations.
- Entrée to linear algebra, especially eigenvalues and eigenvectors.
- Symmetry groups of graphs is the other branch of Algebraic Graph Theory.

Eigenvalues of Graphs

λ is an eigenvalue of a graph
$\Leftrightarrow$ λ is an eigenvalue of the adjacency matrix
$\Leftrightarrow$ $A\vec{x} = \lambda\vec{x}$ for some vector $\vec{x}$

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Adjacency matrix is real, symmetric $\Rightarrow$
real eigenvalues, algebraic and geometric multiplicities are equal
minimal polynomial is product of linear factors for distinct eigenvalues

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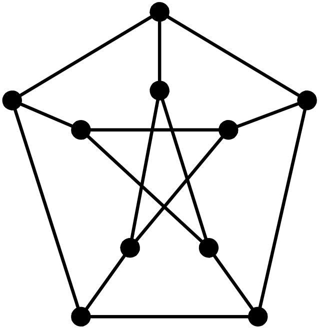
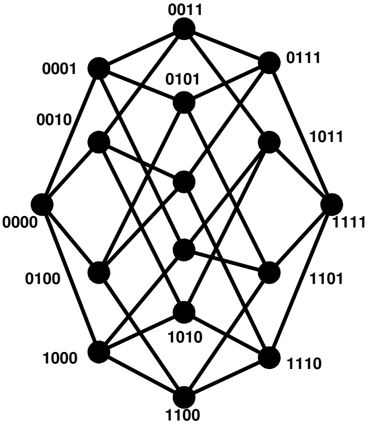


Eigenvalues:

$\lambda = 3$	$m = 1$
$\lambda = 1$	$m = 5$
$\lambda = -2$	$m = 4$

4-Dimensional Cube

- Vertices: Length 4 binary strings
- Join strings differing in exactly one bit
- Generalizes 3-D cube



Eigenvalues:

$$\lambda = 4 \quad \lambda = 2 \quad \lambda = 0 \quad \lambda = -2 \quad \lambda = -4$$

$$m = 1 \quad m = 4 \quad m = 6 \quad m = 4 \quad m = 1$$

Regular Graphs

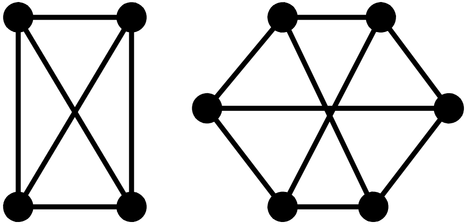
A graph is regular if every vertex has the same number of edges incident. The degree is the common number of incident edges.

Theorem

Suppose G is a regular graph of degree r. Then

- *r is an eigenvalue of G*
- *The multiplicity of r is the number of connected components of G*

Regular of degree 3 with 2 components implies that $\lambda = 3$ will be an eigenvalue of multiplicity 2.



Proof.

- Let $\vec{u}$ be the vector where every entry is 1. Then

$$A\vec{u} = A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} r \\ r \\ \vdots \\ r \end{bmatrix} = r\vec{u}$$

- For each component of the graph, form a vector with 1's in entries corresponding to the vertices of the component, and zeros elsewhere.

  These eigenvectors form a basis for the eigenspace of r.
  (Their sum is the vector $\vec{u}$ above.)

$\square$

Labeling Puzzles Explained

The product of a graph's adjacency matrix with a column vector, $A\vec{u}$, forms sums of entries of $\vec{u}$ for all adjacent vertices.

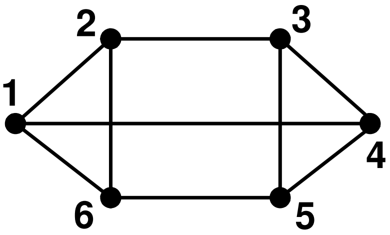
If $\vec{u}$ is an eigenvector, then these sums should equal a common multiple of the numbers assigned to each vertex. This multiple is the eigenvalue.

So the puzzles earlier were simply asking for:

- eigenvectors (assignments of numbers to vertices), and
- eigenvalues (common multiples)

of the adjacency matrix of the graph.

Eigenvalues and Eigenvectors of the Prism



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

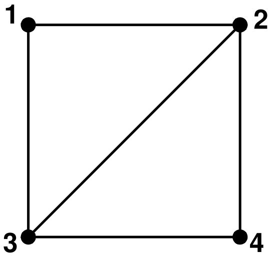
$\lambda = 3$
$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 1$
$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = 0$
$$\begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = -2$
$$\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigenvalues and Eigenvectors



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\lambda = -1 \qquad \lambda = 0 \qquad \lambda = \frac{1}{2}\left(\sqrt{17}+1\right) \qquad \lambda = \frac{1}{2}\left(-\sqrt{17}+1\right)$$

$$\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{17}-1 \\ 4 \\ \sqrt{17}-1 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{17}+1 \\ -4 \\ \sqrt{17}+1 \\ -4 \end{bmatrix}$$

Powers of Adjacency Matrices

Theorem

Suppose A is the adjacency matrix of a graph. Then the number of walks of length ℓ between vertices v_i and v_j is the entry in row i, column j of A^ℓ.

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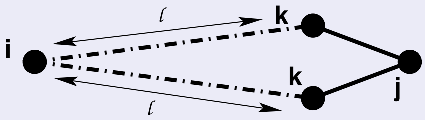
Proof.

Base case: $\ell = 1$. Adjacency matrix describes walks of length 1.

$$\left[A^{\ell+1}\right]_{ij} = \left[A^\ell A\right]_{ij} = \sum_{k=1}^{n} \left[A^\ell\right]_{ik} [A]_{kj} = \sum_{k:v_k \text{ adj } v_j} \left[A^\ell\right]_{ik}$$

$= \text{total ways to walk in } \ell \text{ steps from } v_i \text{ to } v_k\text{, a neighbor of } v_j$



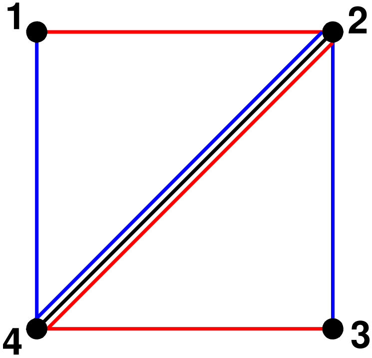
□

Adjacency Matrix Power

Number of walks of length 3 between vertices 1 and 3?

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 5 & \boxed{2} & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{bmatrix}$$

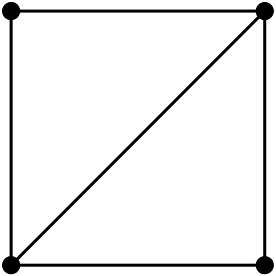


Characteristic Polynomial

Recall the characteristic polynomial of a square matrix A is $\det(\lambda I - A)$. Roots of this polynomial are the eigenvalues of the matrix.

For the adjacency matrix of a graph on n vertices:

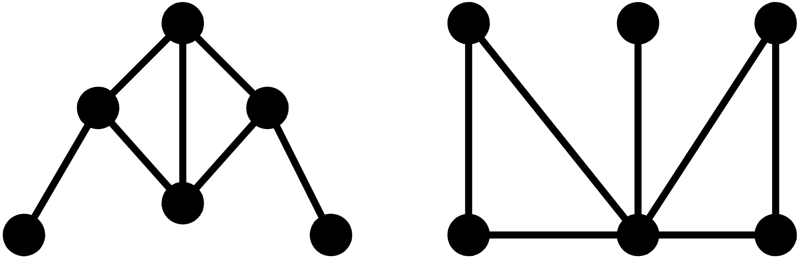
- Coefficient of λ^n is 1
- Coefficient of λ^{n-1} is zero (trace of adjacency matrix)
- Coefficient of $-\lambda^{n-2}$ is number of edges
- Coefficient of $-2\lambda^{n-3}$ is number of triangles



Characteristic polynomial
$\lambda^4 - 5\lambda^2 - 4\lambda + \cdots$

Cospectral Graphs

- Eigenvalues do not characterize graphs!
- Knowing the eigenvalues does not tell you which graph you have



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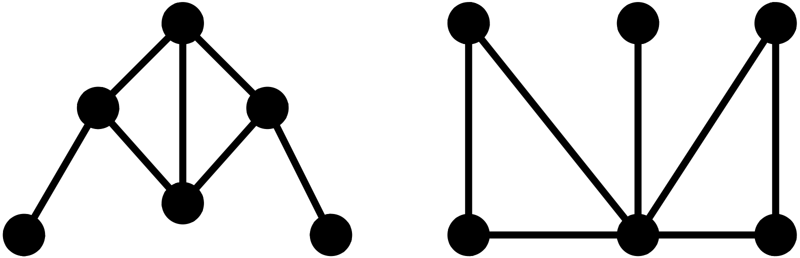


- Two characteristic polynomials of degree 6 are equal
- 4 of the 7 coefficients obviously equal

Diameter and Eigenvalues

The diameter of a graph is the longest shortest path.
In other words, find the shortest route between each pair of vertices, then ask which pair is farthest apart?

Theorem

Suppose G is a graph of diameter d.
Then G has at least $d+1$ distinct eigenvalues.

Proof.

Minimal polynomial is product of linear factors, one per distinct eigenvalue.

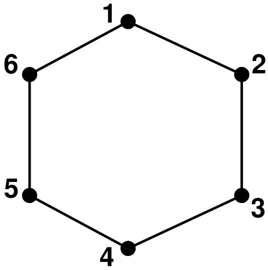
There are zero short walks between vertices that are far apart.

So the first d powers of A are linearly independent.

So A cannot satisfy a polynomial of degree d or less.

Thus minimal polynomial has at least $d+1$ factors
hence at least $d+1$ eigenvalues. □

Minimal Polynomial for a Diameter 3 Graph



Distinct Eigenvalues: $\lambda = 2,\ 1,\ -1,\ -2$

Minimal Polynomial:
$(x-2)(x-1)(x-(-1))(x-(-2)) = x^4 - 5x^2 + 4$

Check that $A^4 - 5A^2 + 4 = 0$

A

$$\begin{bmatrix} 0 & 1 & 0 & \boxed{0} & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

A^2

$$\begin{bmatrix} 2 & 0 & 1 & \boxed{0} & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

A^3

$$\begin{bmatrix} 0 & 3 & 0 & \boxed{2} & 0 & 3 \\ 3 & 0 & 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 3 & 0 & 2 \\ 2 & 0 & 3 & 0 & 3 & 0 \\ 0 & 2 & 0 & 3 & 0 & 3 \\ 3 & 0 & 2 & 0 & 3 & 0 \end{bmatrix}$$

Results Relating Diameters and Eigenvalues

Regular graph: n vertices, degree r, diameter d
Let λ denote second largest eigenvalue (r is largest eigenvalue)

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- Alon, Milman, 1985

$$d \leq 2 \left\lceil \left(\frac{2r}{r - \lambda} \right)^{\frac{1}{2}} \log_2 n \right\rceil$$

Results Relating Diameters and Eigenvalues

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$$d \leq 2\left\lceil \left(\frac{2r}{r-\lambda}\right)^{\frac{1}{2}} \log_2 n \right\rceil$$

- Mohar, 1991

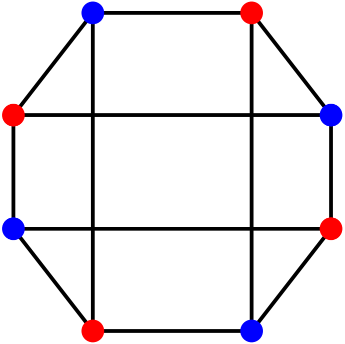
$$d \leq 2\left\lceil \left(\frac{2r-\lambda}{4\,(r-\lambda)}\right) \ln\left(n-1\right) \right\rceil$$

Bipartite Graphs

A graph is bipartite if the vertex set can be split into two parts, so that every edge goes from part to the other. Identical to being “two-colorable.”

Theorem

A graph is bipartite if and only if there are no cycles of odd length.



Eigenvalues of Bipartite Graphs

Theorem

Suppose G is a bipartite graph with eigenvalue λ. Then $-\lambda$ is also an eigenvalue of G.

$$\lambda = 3 \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \lambda = -3 \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad \lambda = 1 \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \lambda = -1 \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

Proof.

Order the vertices according to the two parts. Then

$$A = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix}$$

For an eigenvector $\vec{u} = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix}$ of A for λ,

$$\begin{bmatrix} \lambda \vec{u}_1 \\ \lambda \vec{u}_2 \end{bmatrix} = \lambda \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = \lambda \vec{u} = A\vec{u} = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = \begin{bmatrix} B\vec{u}_2 \\ B^t\vec{u}_1 \end{bmatrix}$$

Now set $\vec{v} = \begin{bmatrix} -\vec{u}_1 \\ \vec{u}_2 \end{bmatrix}$ and compute,

$$A\vec{v} = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix} \begin{bmatrix} -\vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = \begin{bmatrix} B\vec{u}_2 \\ -B^t\vec{u}_1 \end{bmatrix} = \begin{bmatrix} \lambda \vec{u}_1 \\ -\lambda \vec{u}_2 \end{bmatrix} = -\lambda \begin{bmatrix} -\vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = -\lambda \vec{v}$$

□

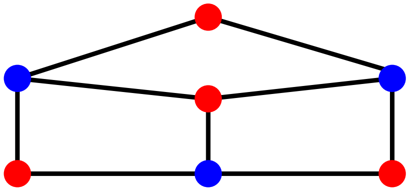
Bipartite Graphs on an Odd Number of Vertices

Theorem

Suppose G is a bipartite graph with an odd number of vertices. Then zero is an eigenvalue of the adjacency matrix of G.

Proof.

Pair each eigenvalue with its negative.
At least one eigenvalue must then equal its negative. $\square$



$\lambda = \sqrt{7}$	$m = 1$
$\lambda = 1$	$m = 2$
$\lambda = 0$	$m = 1$
$\lambda = -1$	$m = 2$
$\lambda = -\sqrt{7}$	$m = 1$

Application: Friendship Theorem

Theorem

Suppose that at a party every pair of guests has exactly one friend in common. Then there are an odd number of guests, and one guest knows everybody.

- Model party with a graph
- Vertices are guests, edges are friends
- Two possibilities:

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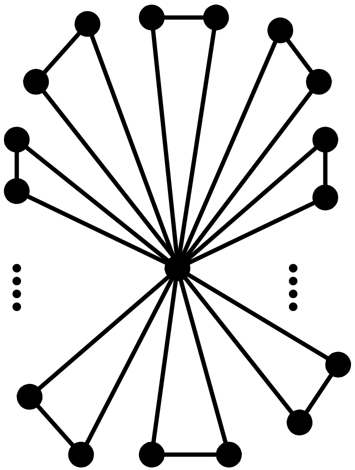

- Model party with a graph
- Vertices are guests, edges are friends
- Two possibilities:
  1. Graph to the right
  2. Algebraic graph theory say the adjacency matrix has eigenvalues with non-integer multiplicities

Application: Buckministerfullerene

- Molecule composed solely of 60 carbon atoms
- "Carbon-60," "buckyball"
- At each atom two single bonds and one double bond describe a graph that is regular of degree 3
- Graph is the skeleton of a truncated icosahedron, a soccer ball
- 60×60 adjacency matrix
- Eigenvalues of adjacency matrix predict peaks in mass spectrometry
- First created in 1985 (1996 Nobel Prize in Chemistry)

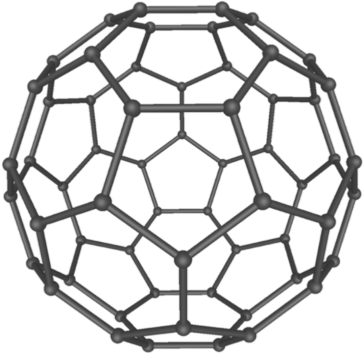

Illustration by Michael Ströck
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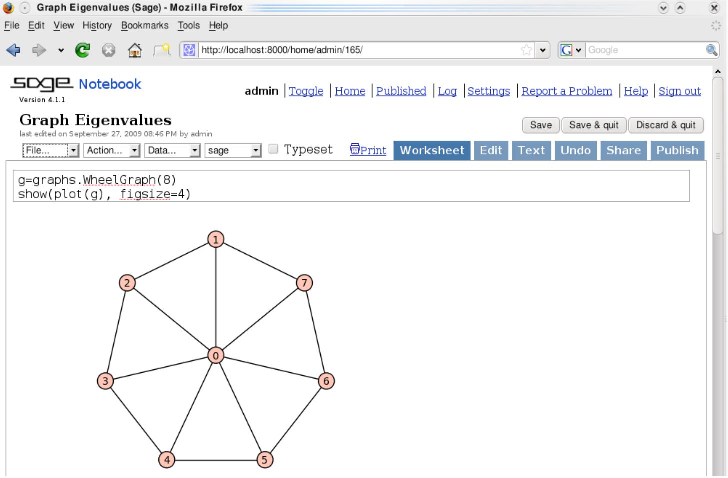
Sage and Graphs

Sage is software for mathematics, “creating a viable free open source alternative to Magma, Maple, Mathematica and Matlab.”



Sage and Graph Eigenvalues

```
print g.spectrum()
print g.eigenspaces()
```

```
[3.828427124746190?, 1.246979603717467?, 1.246979603717467?,
-0.4450418679126288?, -0.4450418679126288?, -1.801937735804839?,
-1.801937735804839?, -1.828427124746191?]
[
(a0, Vector space of degree 8 and dimension 1 over Number Field in a0
with defining polynomial x^2 - 2*x - 7
User basis matrix:
[     1 1/7*a0 1/7*a0 1/7*a0 1/7*a0 1/7*a0 1/7*a0 1/7*a0]),
(a1, Vector space of degree 8 and dimension 2 over Number Field in a1
with defining polynomial x^3 + x^2 - 2*x - 1
User basis matrix:
[        0          1          0         -1        -a1 -a1^2 + 1  a1^2 - 1
a1]
[        0          0          1         a1  a1^2 - 1 -a1^2 + 1        -a1
-1])
]
```

```
f(x)=x^2 - 2*x - 7
g(x)=x^3 + x^2 - 2*x - 1
print f(-1.828427124746191)
print g(1.246979603717467)
```

```
4.88498130835069e-15
-4.44089209850063e-16
```

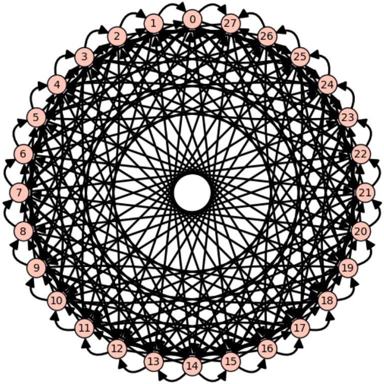
Isospectral, Non-Isomorphic Circulant Graphs

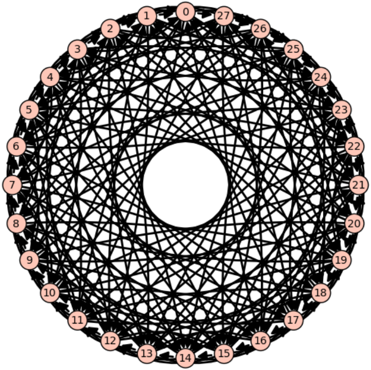
- A graph is circulant if each row of the adjacency matrix is a shift (by one) of the prior row.

Isospectral, Non-Isomorphic Circulant Graphs

- A graph is circulant if each row of the adjacency matrix is a shift (by one) of the prior row.
- Infinite family of pairs of digraphs:
  identical spectra, but non-isomorphic graphs
- Vertices number $n = 2^r p$, with $r \geq 2$, p prime
- Julia Brown, 2008 Reed College thesis

$n = 2^2 \cdot 7$





Sage: 6 lines to construct, 2 lines to plot, 2 lines to verify
Also verified for $n = 2^5 \cdot 13 = 416$ (in under 6 seconds)

Moral of the Story

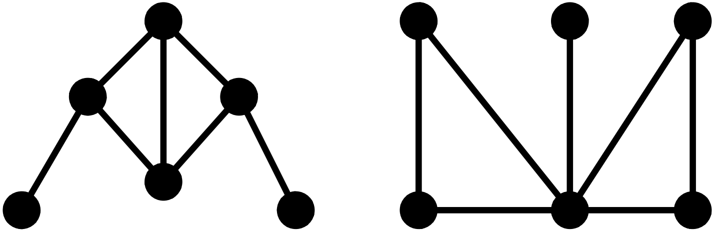
- Graph properties predict linear algebra properties
  (regular of degree $r \Rightarrow r$ is an eigenvalue)

Moral of the Story

- Graph properties predict linear algebra properties
  (regular of degree $r \Rightarrow r$ is an eigenvalue)
- Linear algebra properties predict graph properties
  (functions of two largest eigenvalues bound the diameter)

Moral of the Story

- Graph properties predict linear algebra properties
  (regular of degree $r \Rightarrow r$ is an eigenvalue)
- Linear algebra properties predict graph properties
  (functions of two largest eigenvalues bound the diameter)
- Do eigenvalues characterize graphs? No.



Bibliography

- Norman Biggs, *Algebraic Graph Theory*, Second Edition
  Cambridge Mathematical Library, 1974, 1993
- Chris Godsil, Gordon Royle, *Algebraic Graph Theory*
  Springer, 2001
- Julia Brown, *Isomorphic and Nonisomorphic, Isospectral Circulant Graphs*, arXiv:0904.1968v1, 2009
- Sage, `http://sagemath.org`
  Free, public notebook server at `http://sagenb.org`
- Robert Beezer, *A First Course in Linear Algebra*
  `http://linear.ups.edu`, GFDL License

Posted at `http://buzzard.ups.edu/talks.html`