# Everything you Wanted to Know About the Hoffman-Singleton Graph

... but were afraid to draw

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MAA Summer Meeting
Portland, Oregon
August 6, 2009

Available at http://buzzard.ups.edu/talks.html

## Cages

- Girth — length of shortest circuit
- $(r, g)$-cage — smallest regular graph with degree $r$ and girth $g$
- Moore graph — an $(r, g)$-cage meeting obvious lower bound
  - Example: Petersen Graph, the (3, 5)-cage
  - Qualifies as a Moore graph: $1 + 3 + 3(2) = 10$ vertices

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## Steiner Systems

- Block design, notation is $S(\lambda, m, n)$
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## Automorphism Group

- Permutations of vertices "preserving" edges
- $\sigma \in \mathrm{Aut}(G)$ if:
  $(u, v)$ is an edge of $G \iff (\sigma(u), \sigma(v))$ is an edge of $G$

# Construction of HS

## Vertices

- 50 vertices total
- $\mathbf{Z}_5 \times \mathbf{Z}_5$, $(a,\ b)$, $0 \leq a, b < 5$
- Think of these as Cartesian "$x$–$y$" pairs

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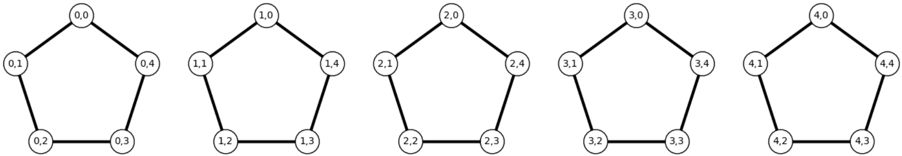
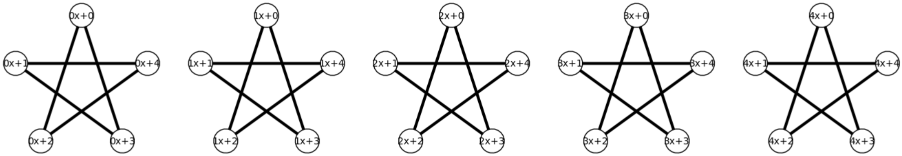
## Edges

- 5 pentagons
- $(a, b)$ adjacent to $(a, b \pm 1) \pmod 5$
- 5 pentagrams
- $mx + c$ adjacent to $mx + (c \pm 2) \pmod 5$

# Pentagons and Pentagrams



# More Edges

- $(a, b)$ is adjacent to $mx + c \iff ma + c = b \pmod 5$
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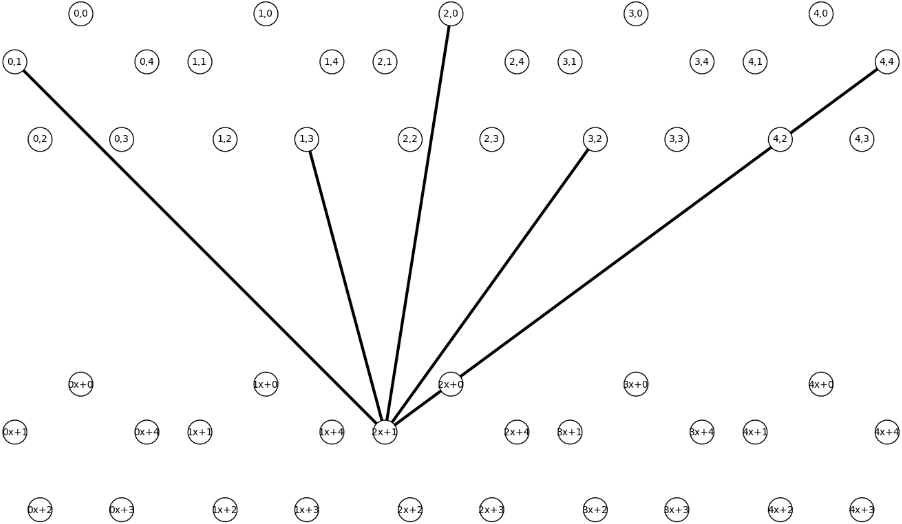
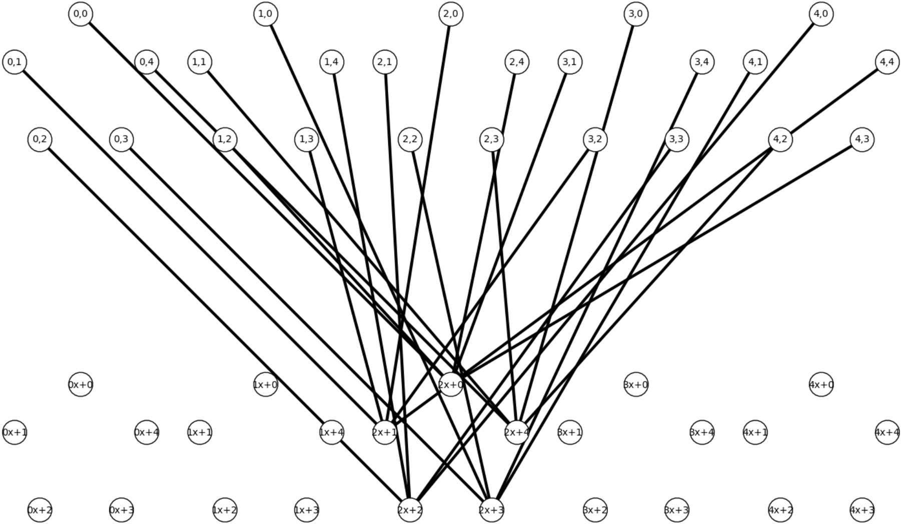
- Regular of degree $2 + 5 = 7$
- $25 \times 5 = 125$ additional edges
- 175 total edges

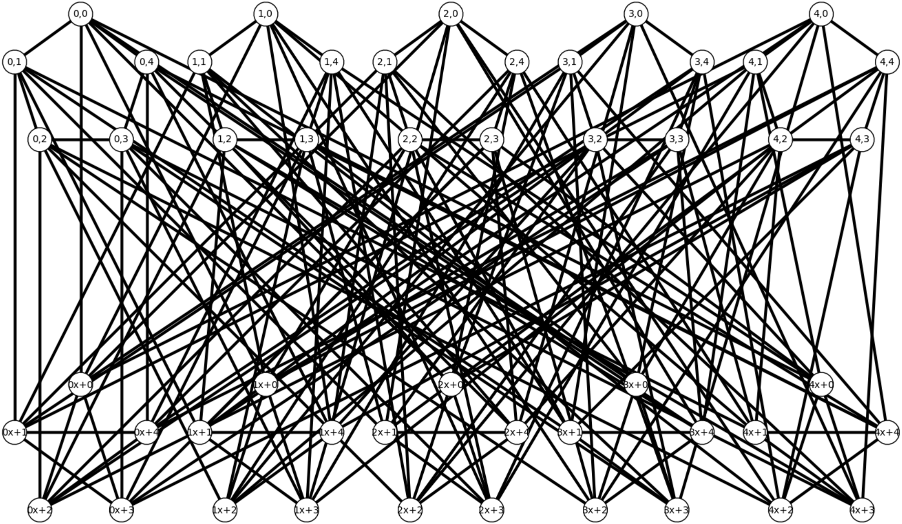
# Neighbors of $2x + 1$



# Neighbors of Slope 2 Pentagram



# The HS graph


0,0
1,0
2,0
3,0
4,0
0,1
0,4
1,1
1,4
2,1
2,4
3,1
3,4
4,1
4,4
0,2
0,3
1,2
1,3
2,2
2,3
3,2
3,3
4,2
4,3
0x+0
1x+0
2x+0
3x+0
4x+0
0x+1
0x+4
1x+1
1x+4
2x+1
2x+4
3x+1
3x+4
4x+1
4x+4
0x+2
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1x+2
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2x+2
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3x+2
3x+3
4x+2
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# Movie One

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- $1 + 7 + 7(6) = 50$ is minimum number of vertices
  - Visualize a 7-ary tree of depth 2
- Qualifies HS as a Moore graph

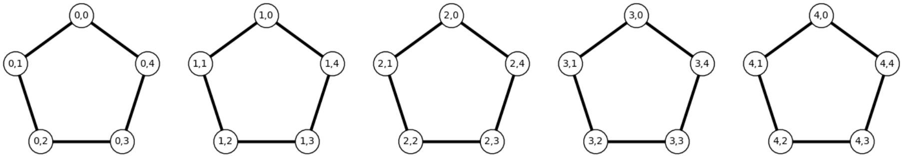
# HS Graph is a Moore Graph

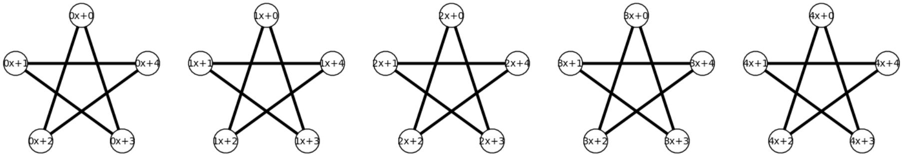
- Smallest circuit has length 5 (pentagons), i.e. girth 5
- Regular of degree 7
- $1 + 7 + 7(6) = 50$ is minimum number of vertices
  - Visualize a 7-ary tree of depth 2
- Qualifies HS as a Moore graph
- Can establish uniqueness

# Pentagonal Structure

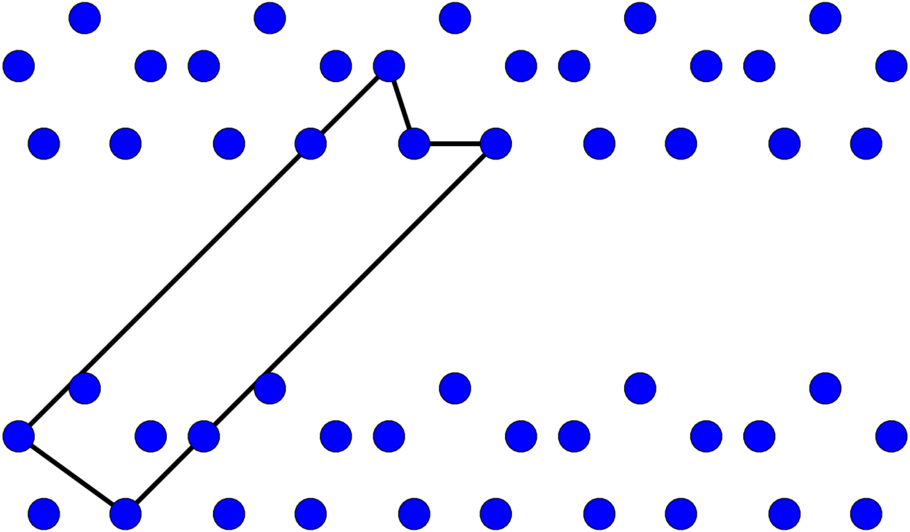
- 5 of type P - P - P - P - P
- 5 of type L - L - L - L - L
- 125 of type P - P - P - L - L
- 125 of type L - L - L - P - P
- 500 of type P - P - L - P - L
- 500 of type L - L - P - L - P
- 1260 pentagons total

# 10 “Base” Pentagons



# P - P - P - L - L



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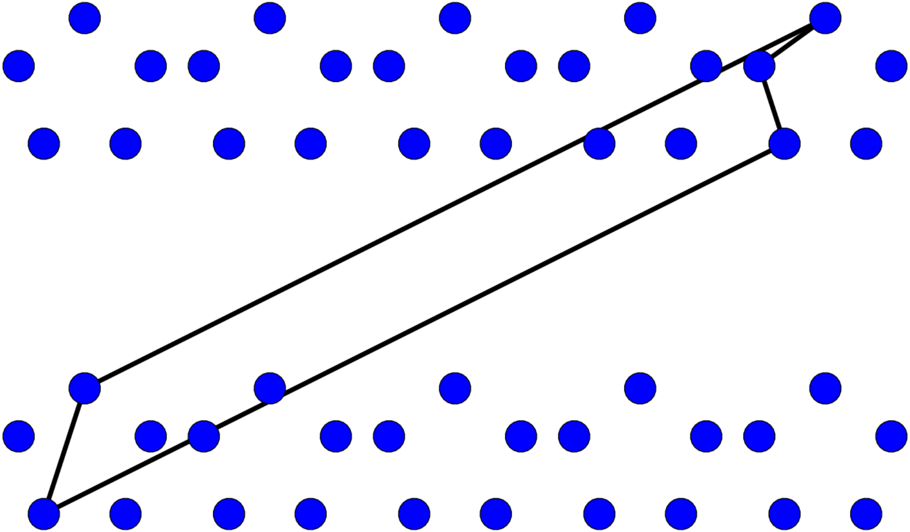
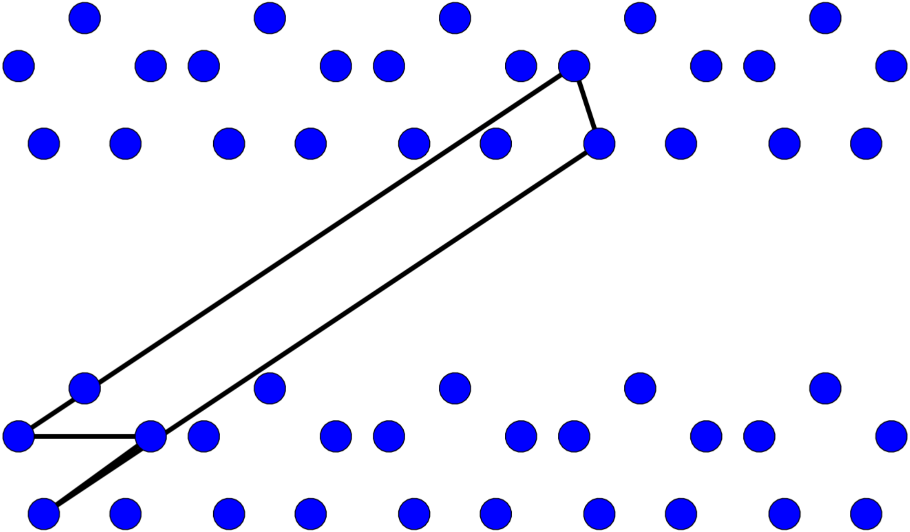


# L - L - L - P - P



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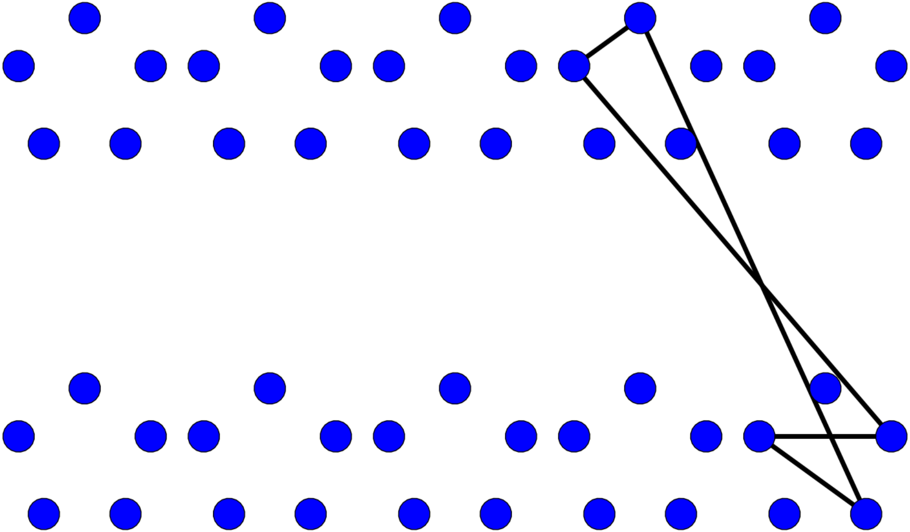

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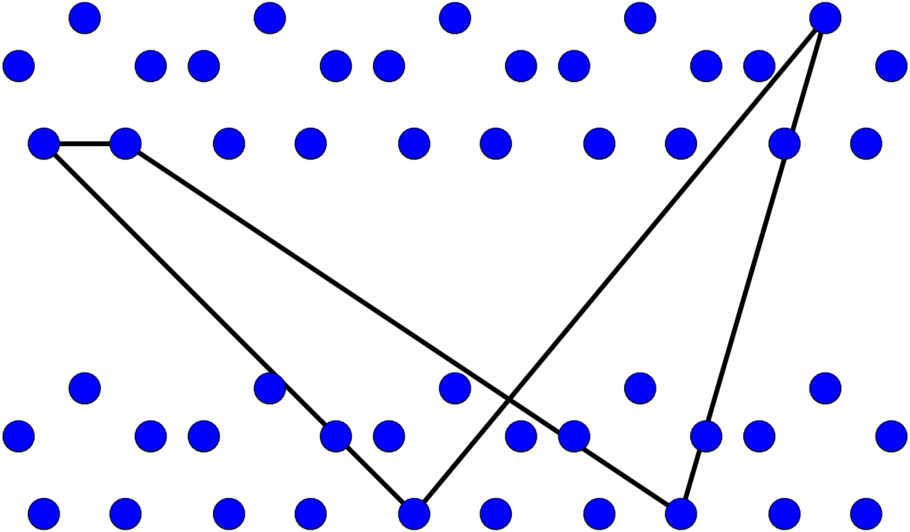

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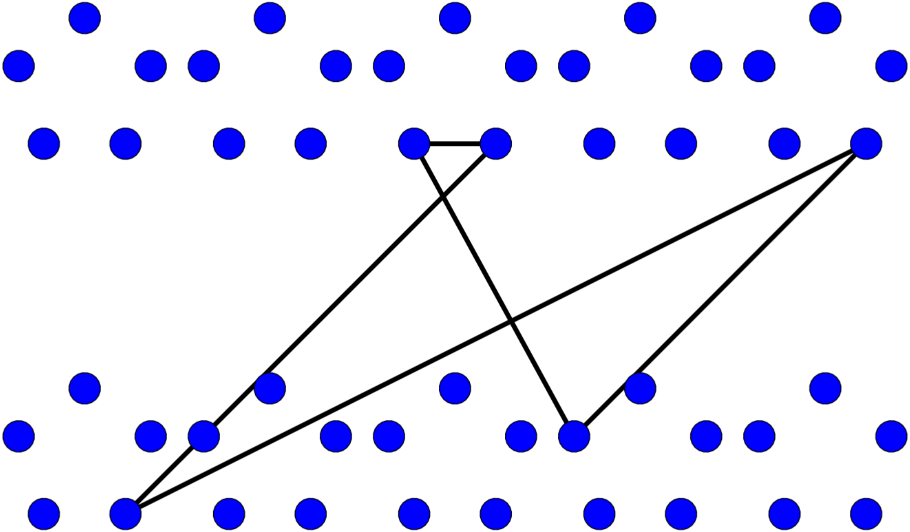

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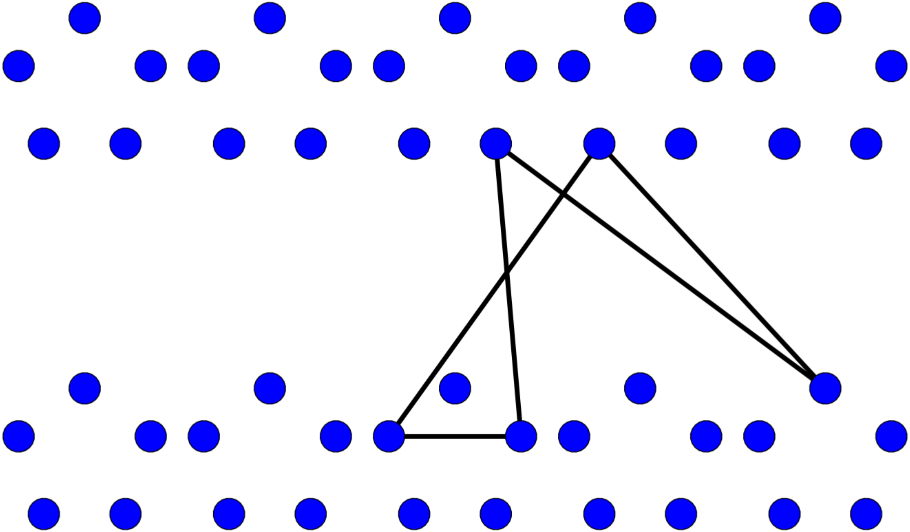

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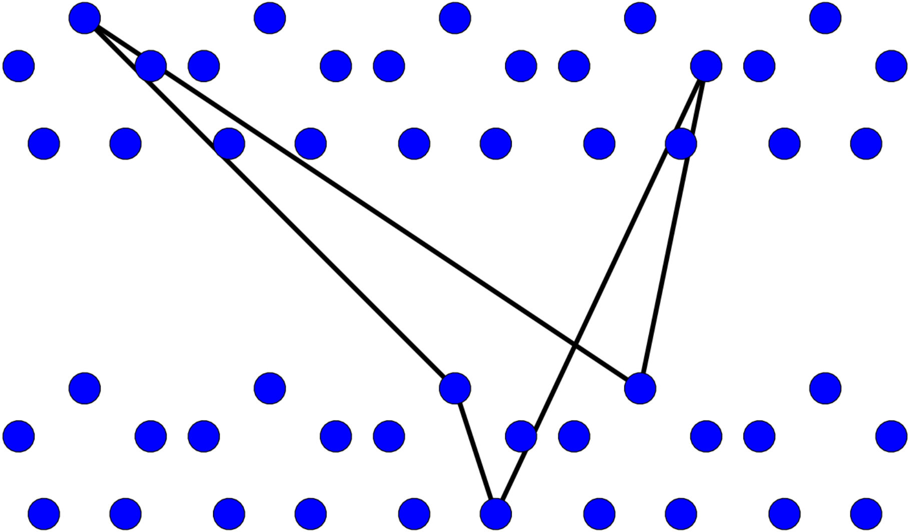
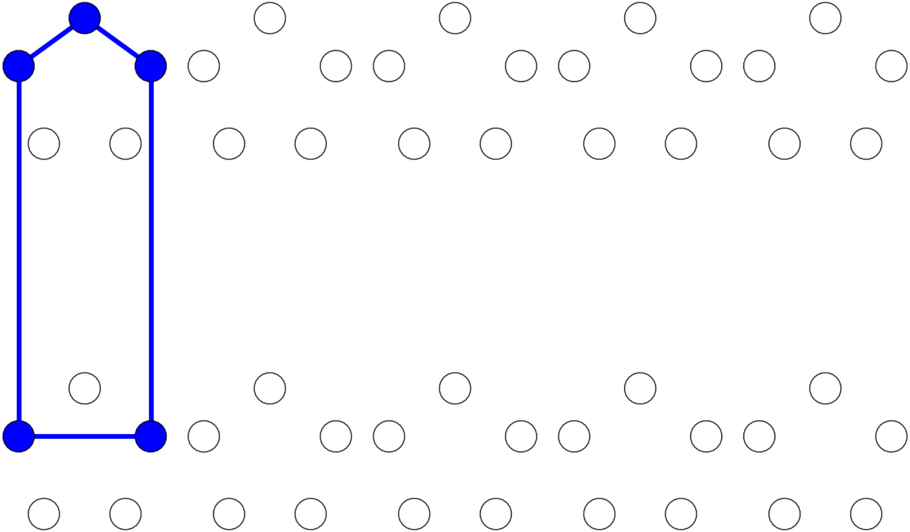


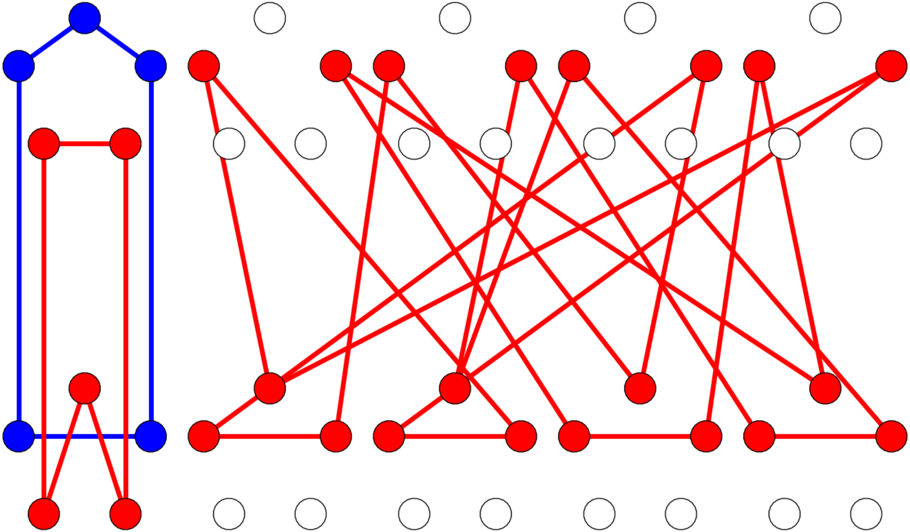
# Collections of Pentagons

- Grab *any one* of the 1260 pentagons
- The five vertices have 25 neighbors
  (apart from the pentagon's own vertices)
- These 25 vertices induce 5 new pentagons
- 20 vertices left, they induce 4 pentagons
- So cover all 50 vertices with 10 pentagons
- "10-pack" of pentagons naturally divides into two "5-packs"

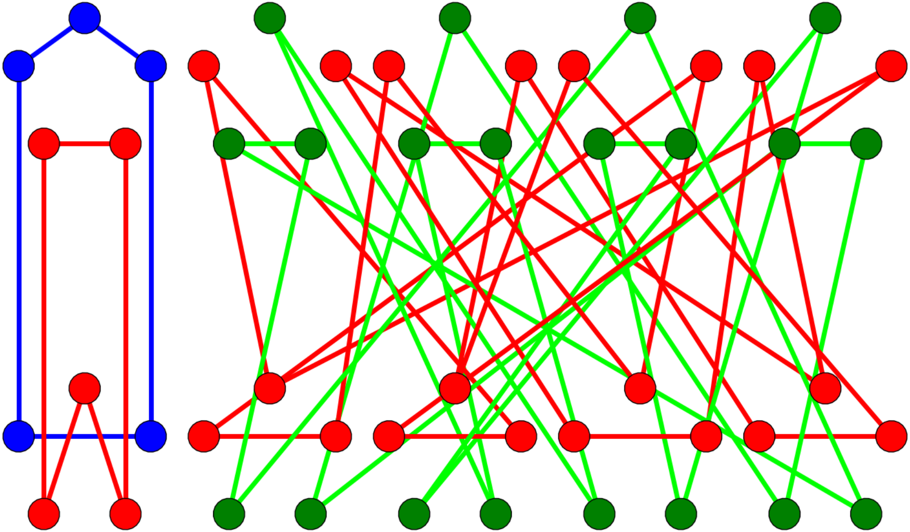
# An Arbitrary Pentagon (Blue)



# 25 Neighbors, 5 New Pentagons (Red)



# 20 Vertices Left, 4 New Pentagons (Green)



# Cages as Subgraphs

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- Choose a 10-pack, remove one pentagon from each 5-pack
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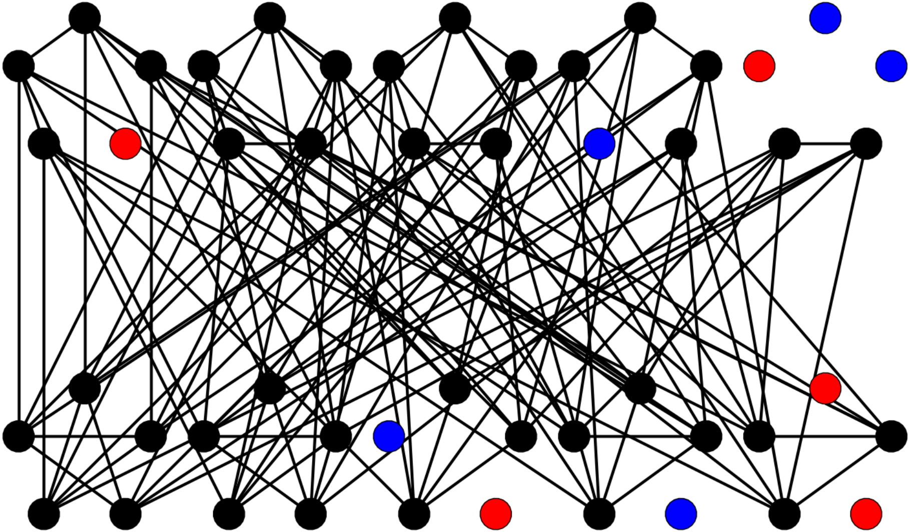
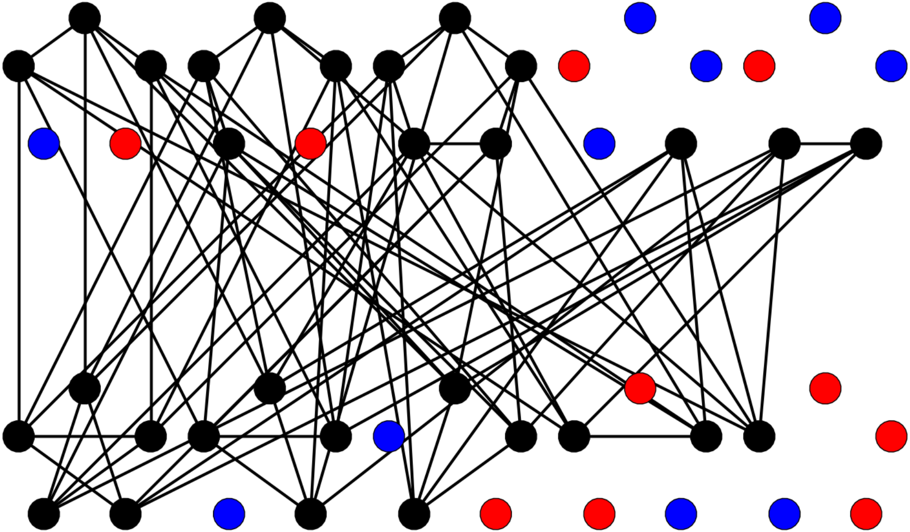
## The (3, 5) cage, i.e. Petersen Graph

- Choose a 10-pack, remove four pentagons from each 5-pack
- Degree 3, 10 vertices, girth 5
- Petersen Graph - 525 copies

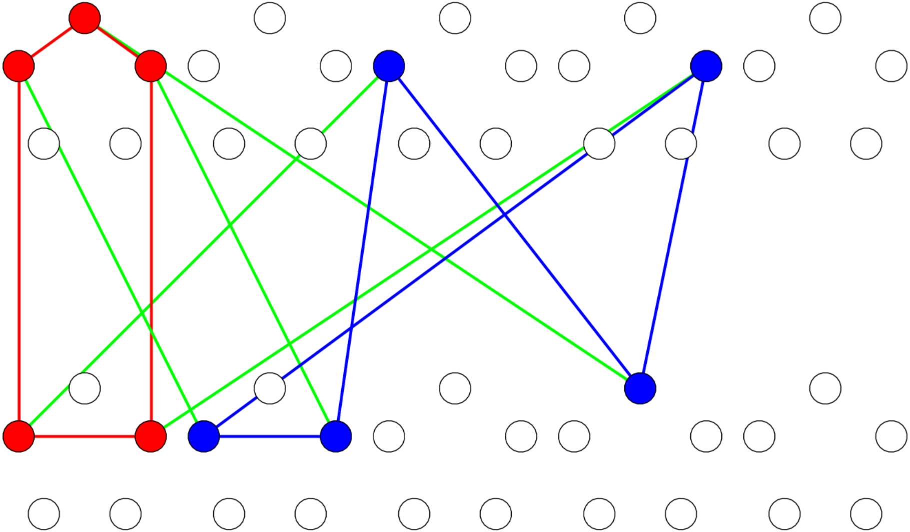
# (6, 5) Cage



## (5, 5) Cage



# (3, 5) Cage



# Another Construction

## Triples

- Given a 7-set, there are $\binom{7}{3} = 35$ possible 3-sets
- Some of these can form the blocks of a $S(2,3,7)$ Steiner system
- Also known as the Fano plane, projective plane of order 2

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## An Example

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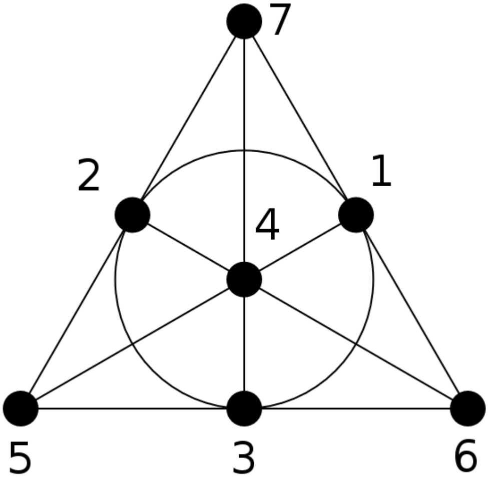
## All Fano Planes

- Thirty ways to do this
- Naturally splits into two sets of 15
  (orbits of $A_7$)

# A Fano Plane



# The HS Graph, Again

## Vertices

- All 3-sets from a 7-set, 35 “triples”
- One natural half of all Fano planes, 15 “planes”

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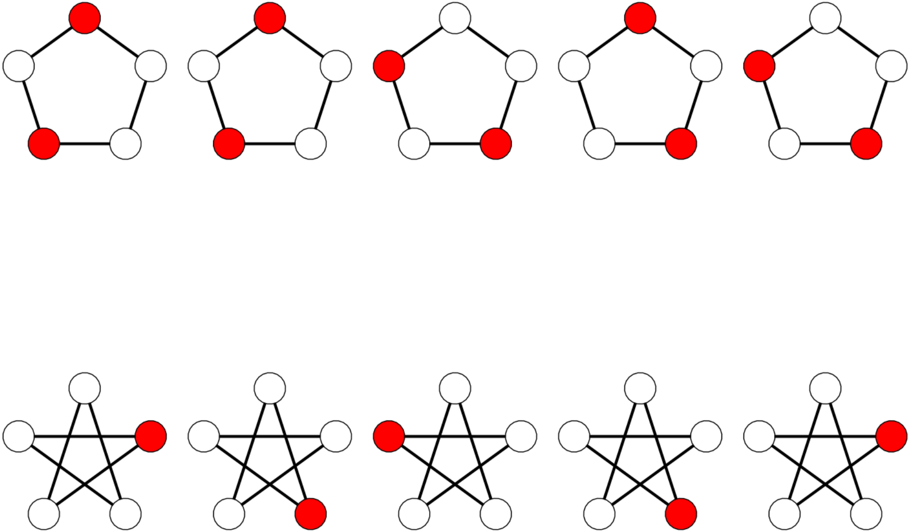
## Edges

- Edge between a triple and a plane $\iff$ triple is part of the plane
- Edge between a triple and another triple $\iff$ disjoint sets
- Edge between a plane and another plane: **never**

# Independent Sets in HS Graph

- Independent sets ("cocliques") are vertex sets with no adjacencies
- Maximum independent set in HS has size 15
- HS has 100 such independent sets, all the "same"

# Maximal Independent Set



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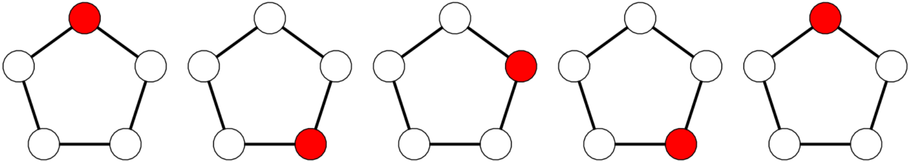

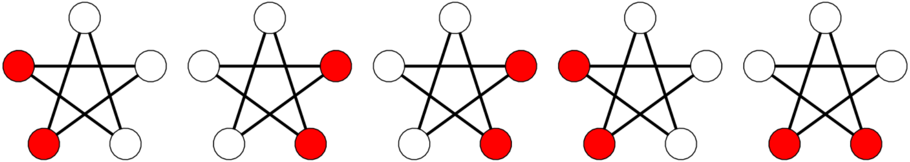

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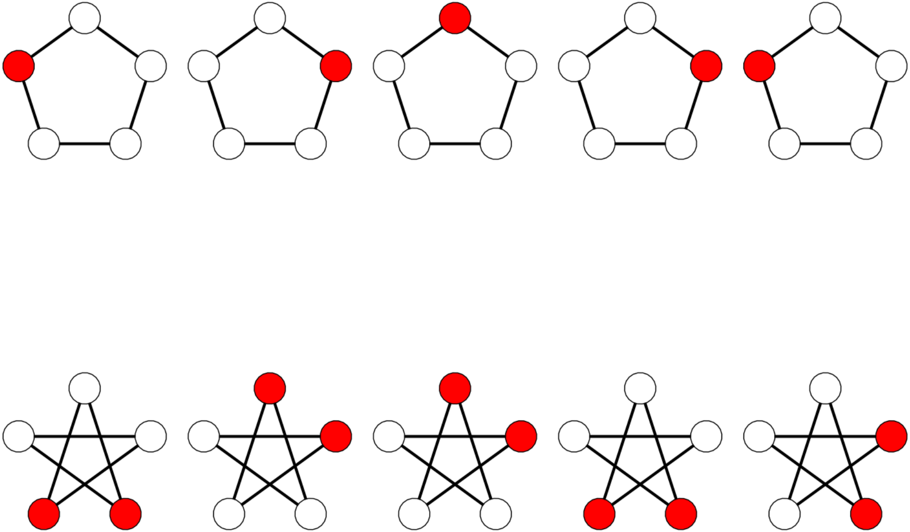

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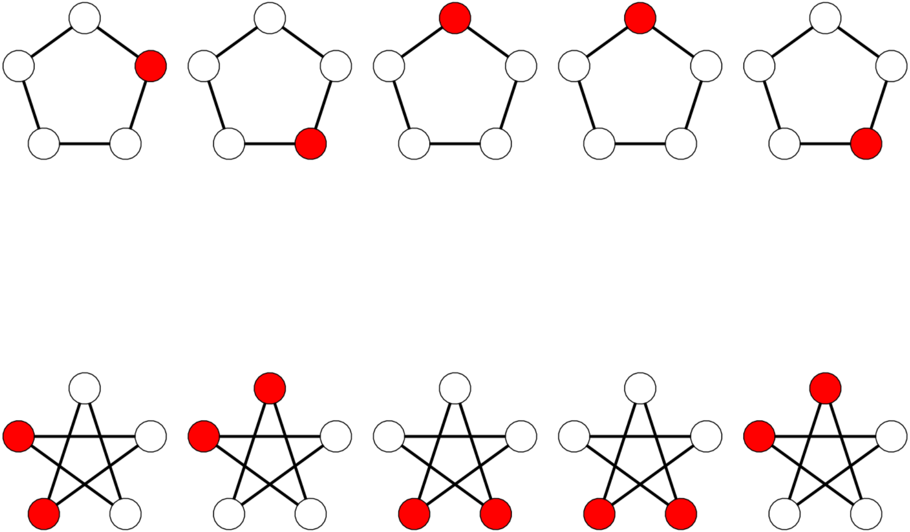
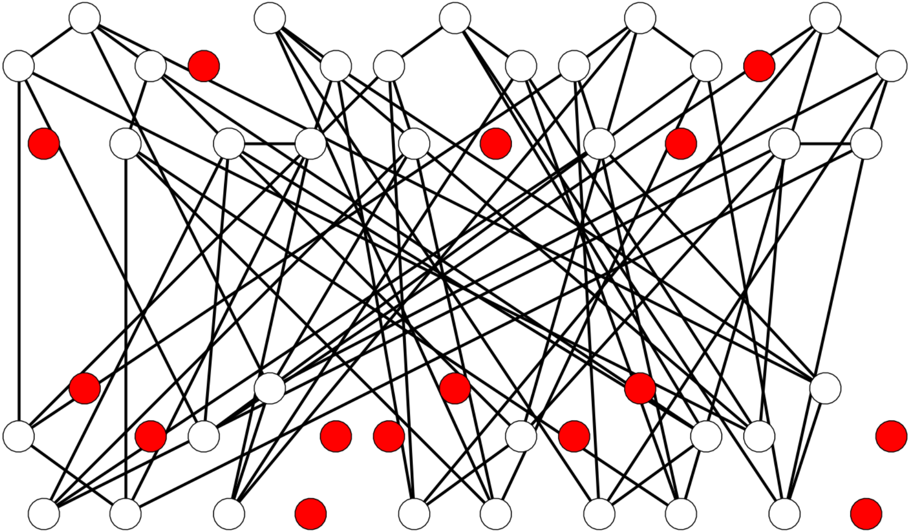


# The Odd Graph $O_4$

- Remove an independent set from HS
- Subgraph induced by remaining vertices is $O_4$
  - 35 vertices, the 3-sets from a 7-set
  - Triples are adjacent if disjoint
- Generally $O_m$ is $m$-sets from a $(2m+1)$-set, with disjoint sets being adjacent
- $O_4$ contains Coxeter graph: degree 3, girth 7, 28 vertices ((3,7)-cage has 24 vertices)

# Odd Graph $O_4$



# Intersections of Independent Sets

## Intersections

- Choose one of the 100 independent sets
- Intersections with the other 99 independent sets are:
  - Empty (7 times)
  - Three vertices (35 times)
  - Five vertices (42 times)
  - Eight vertices (15 times)

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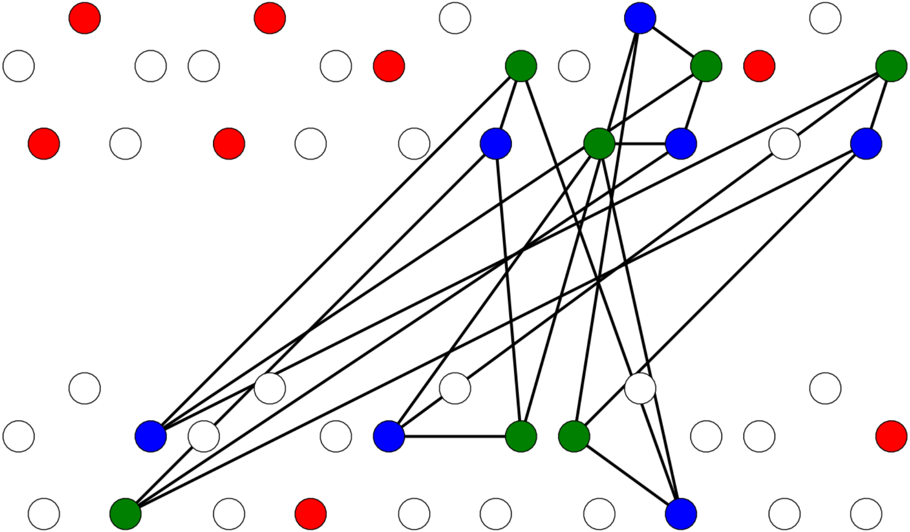
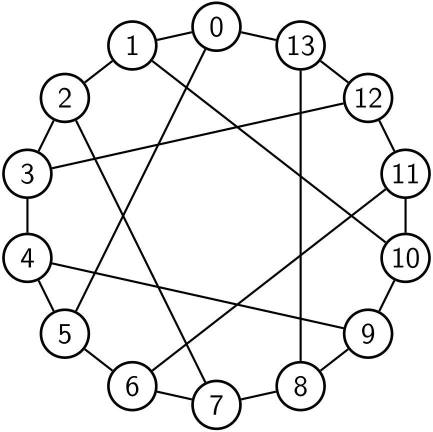
## Two Independent Sets with Maximal Intersection

- Two independent sets meeting in 8 vertices
- Symmetric difference has $7 + 7 = 14$ vertices, bipartite
- Induced subgraph is the Heawood Graph, unique $(3, 6)$-cage
- HS graph has 750 copies of Heawood Graph

# Heawood Graph (Red-Blue Ind Set, Red-Green Ind Set)



# Heawood Graph

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## New Graphs with Independent Sets as Vertices

- Construct a new graph
- Vertices: the 100 independent sets of HS
- Edges: join independent sets that are disjoint
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# New Graphs with Independent Sets as Vertices

- Construct a new graph
- Vertices: the 100 independent sets of HS
- Edges: join independent sets that are disjoint
- Edges: join independent sets that meet in 8 vertices
- This is the Higman-Sims graph
    - Strongly regular, unique for its parameters
    - Easiest construction from Steiner system S(3,6,22)
      1 vertex, 22 symbols, 77 blocks stitched together
    - Splits naturally into two copies of HS (352 ways)
    - Several other interesting subgraphs

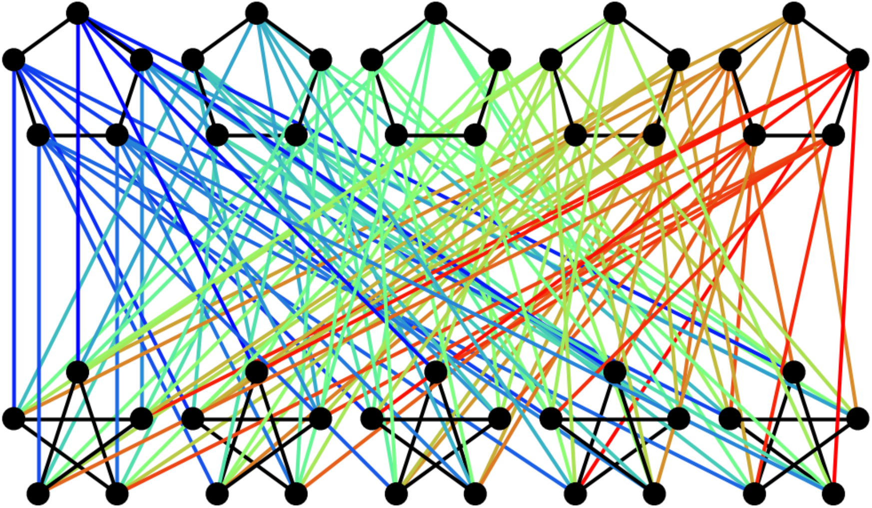
# Automorphisms of the HS Graph

- Permutations of the vertices that take edges to edges
- First, permutations that preserve the 50 edges of the pentagons and pentagrams as a set
- One of order 2, swaps point vertices with line vertices

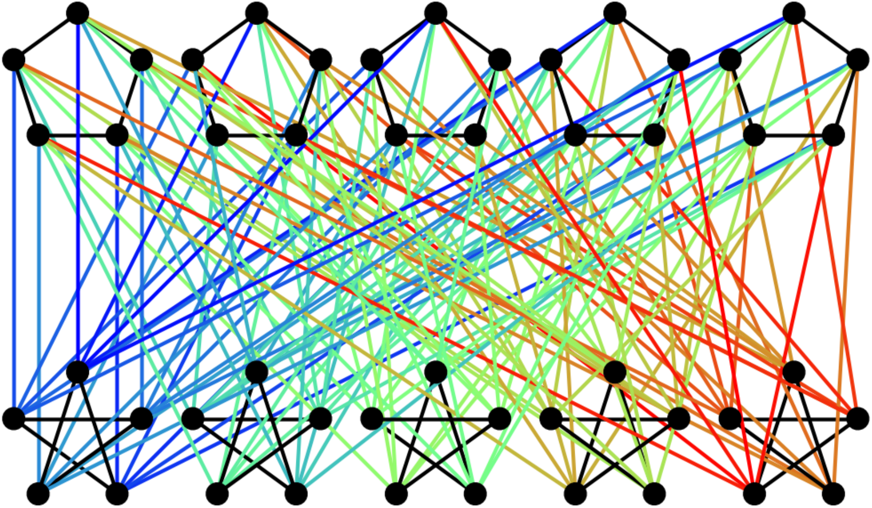
$$(a,\ b) \mapsto (3a)x + (2b)$$

$$mx + c \mapsto (m,\ 2c)$$

# Original with Differentiated Edges



# Order 2 Permutation, Swaps Pentagons & Pentagrams



# Automorphisms of the HS Graph

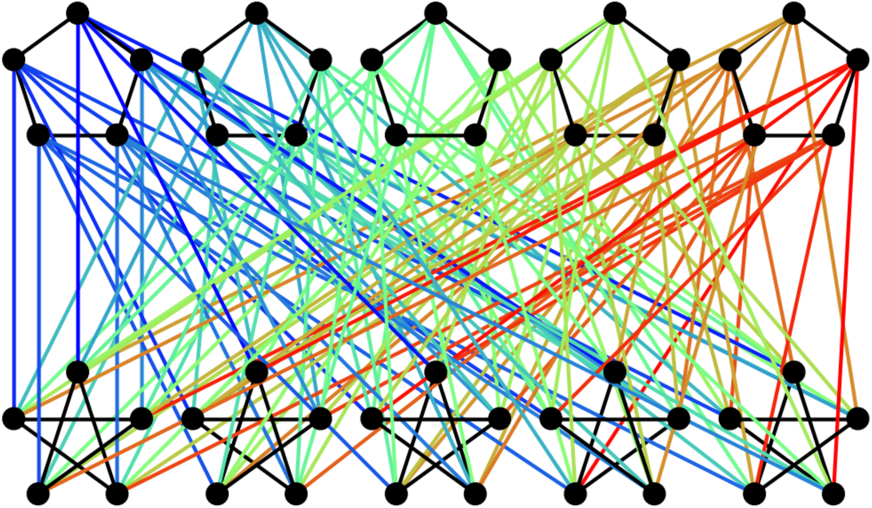
- More permutations that fix the 50 edges of the pentagons and pentagrams as a set
- These fix edges of pentagons and edges of pentagrams as sets
- Parameters: $d \neq 0,\ e = \pm 1,\ f,\ g,\ h \in \mathbf{Z}_5$

$$(a,\ b) \mapsto (da + g,\ fa + eb + h)$$

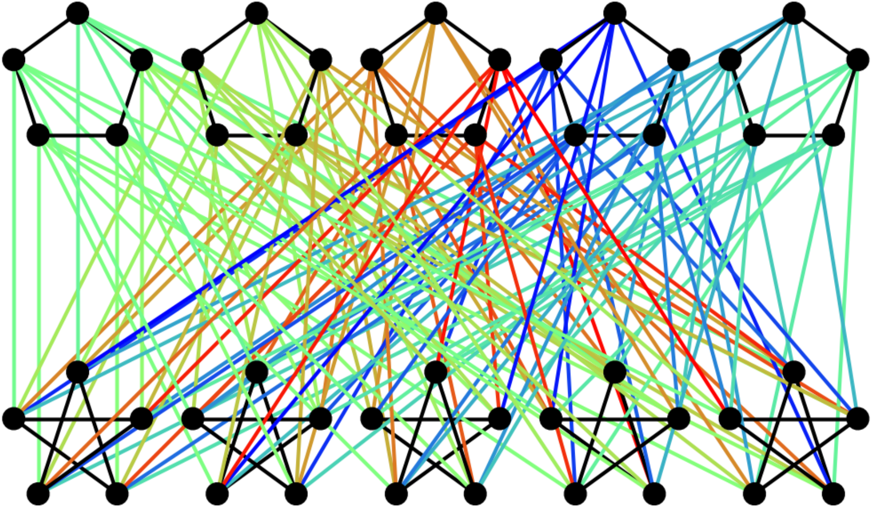
$$mx + c \mapsto \frac{f + em}{d} x + \left( ec + h - \frac{f + em}{d} g \right)$$

- $4 \cdot 2 \cdot 5^3 = 1000$ permutations

# Original HS



# Translation of Pentagons



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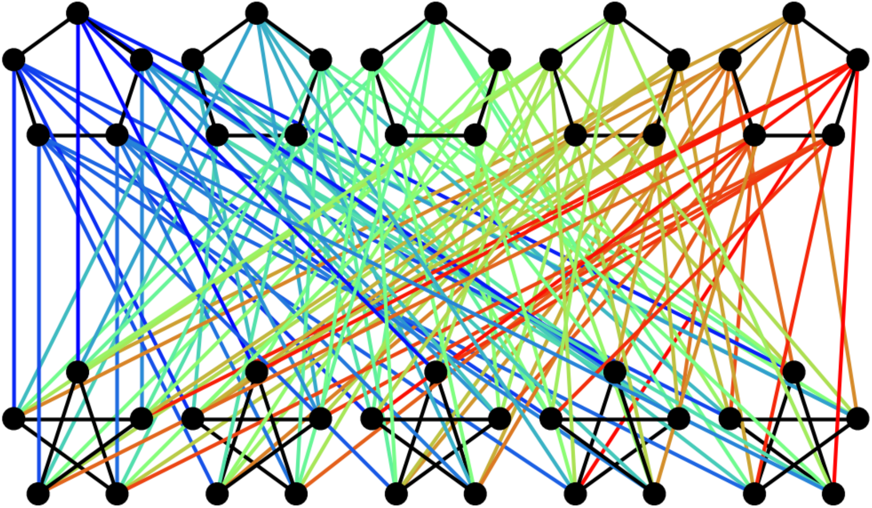
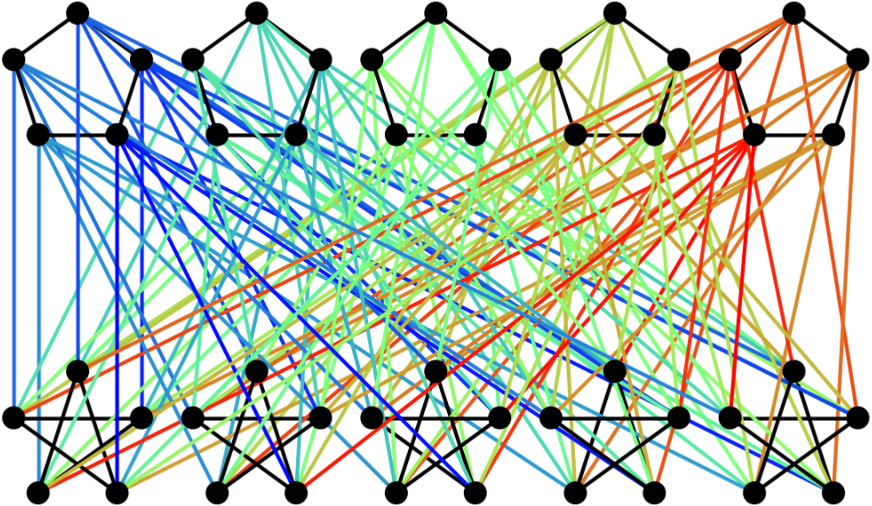


# Rotations on Pentagons



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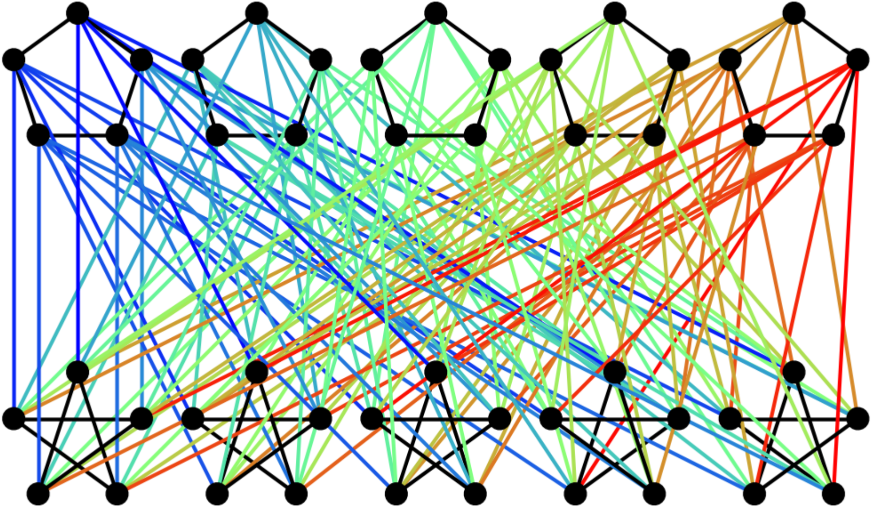
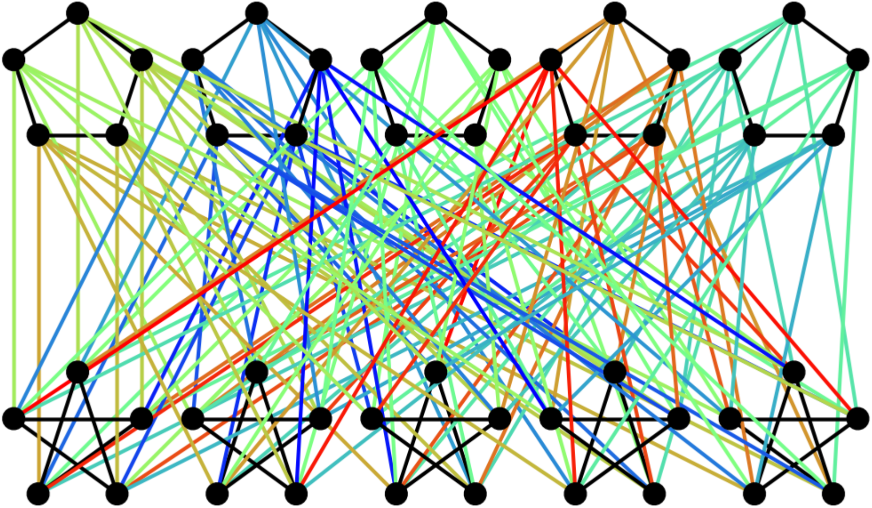


# Total Mixing



# Automorphism Group of HS Graph

- One order 2 permutation, swaps pentagon and pentagram edges
- 1000 permutations fixing pentagon edges and pentagram edges
- Subgroup of order 2000 fixing pentagon and pentagram edges

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- Subgroup of order 2000 fixing pentagon and pentagram edges
- Automorphism group is transitive on the 1260 10-packs of pentagons
- In other words, there exist vertex permutations taking edges of any 10-pack to the edges of any other 10-pack
- Thus, order of automorphism group is

$$2 \cdot 1000 \cdot 1260 = 252,000$$

# Movie Two

# Miscellaneous Facts About HS

- Strongly regular, unique for its parameters
- Hamiltonian
- Line graph is distance-regular
- Vertex-transitive
- Symmetric (transitive on edges, as ordered pairs of vertices)
- Connections to finite geometries
- Connections to codes

# References


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