Everything you Wanted to Know About the Hoffman-Singleton Graph
... but were afraid to draw

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Cages

- Girth — length of shortest circuit
- \((r, g)\)-cage — smallest regular graph with degree \(r\) and girth \(g\)
- Moore graph — an \((r, g)\)-cage meeting obvious lower bound
  - Example: Petersen Graph, the \((3, 5)\)-cage
  - Qualifies as a Moore graph: \(1 + 3 + 3(2) = 10\) vertices
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Steiner Systems

- Block design, notation is \(S(\lambda, m, n)\)
- Collection of \(m\)-sets chosen from an \(n\)-set, “block”
- Every \(\lambda\)-set is in exactly one set of the collection
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Automorphism Group

- Permutations of vertices “preserving” edges
- \(\sigma \in \text{Aut}(G)\) if:
  - \((u, v)\) is an edge of \(G\) \(\iff\) \((\sigma(u), \sigma(v))\) is an edge of \(G\)
Construction of HS

Vertices

- 50 vertices total
- $\mathbb{Z}_5 \times \mathbb{Z}_5$, $(a, b), 0 \leq a, b < 5$
- Think of these as Cartesian “x–y” pairs
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- \( \mathbb{Z}_5 \times \mathbb{Z}_5, mx + c, 0 \leq m, c < 5 \)
- Think of these as lines of slope \( m \), intercept \( c \)
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Edges

- 5 pentagons
- $(a, b)$ adjacent to $(a, b \pm 1)$ (mod 5)
- 5 pentagrams
- $mx + c$ adjacent to $mx + (c \pm 2)$ (mod 5)
Pentagons and Pentagrams
More Edges

- \((a, b)\) is adjacent to \(mx + c \iff ma + c = b \pmod{5}\)
- Each pentagon vertex adjacent to 5 pentagram vertices
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More Edges

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- Each pentagon vertex adjacent to 5 pentagram vertices
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- Regular of degree $2 + 5 = 7$
- $25 \times 5 = 125$ additional edges
- 175 total edges
Neighbors of $2x + 1$
Neighbors of Slope 2 Pentagram
The HS graph
Movie One
HS Graph is a Moore Graph

- Smallest circuit has length 5 (pentagons), i.e. girth 5
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- \(1 + 7 + 7(6) = 50\) is minimum number of vertices
  - Visualize a 7-ary tree of depth 2
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- Smallest circuit has length 5 (pentagons), i.e. girth 5
- Regular of degree 7
- $1 + 7 + 7(6) = 50$ is minimum number of vertices
  - Visualize a 7-ary tree of depth 2
- Qualifies HS as a Moore graph
- Can establish uniqueness
Pentagonal Structure

- 5 of type P - P - P - P - P
- 5 of type L - L - L - L - L
- 125 of type P - P - P - L - L
- 125 of type L - L - L - P - P
- 500 of type P - P - L - P - L
- 500 of type L - L - P - L - P
- 1260 pentagons total
10 “Base” Pentagons
L - L - L - P - P
P - P - L - P - L
Hoffman-Singleton Graph
L - L - P - L - P
Collections of Pentagons

- Grab *any one* of the 1260 pentagons
- The five vertices have 25 neighbors (apart from the pentagon’s own vertices)
- These 25 vertices induce 5 new pentagons
- 20 vertices left, they induce 4 pentagons
- So cover all 50 vertices with 10 pentagons
- “10-pack” of pentagons naturally divides into two “5-packs”
An Arbitrary Pentagon (Blue)
25 Neighbors, 5 New Pentagons (Red)
20 Vertices Left, 4 New Pentagons (Green)
Cages as Subgraphs

The (6, 5) cage

- Choose a 10-pack, remove one pentagon from each 5-pack
- Degree 6, 40 vertices, girth 5, unique

The (5, 5) cage

- Choose a 10-pack, remove two pentagons from each 5-pack
- Degree 5, 30 vertices, girth 5, one of four possible types

The (4, 5) cage – Not

The (4, 5) cage is the Robertson graph

- Robertson graph has 19 vertices, not 20

The (3, 5) cage, i.e. Petersen Graph

- Choose a 10-pack, remove four pentagons from each 5-pack
- Degree 3, 10 vertices, girth 5

Petersen Graph – 525 copies
Cages as Subgraphs

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(6, 5) Cage
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Another Construction

Triples

- Given a 7-set, there are \( \binom{7}{3} = 35 \) possible 3-sets
- Some of these can form the blocks of a \( S(2, 3, 7) \) Steiner system
- Also known as the Fano plane, projective plane of order 2
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An Example

- 123 356 257 145 347 246 167
Another Construction

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All Fano Planes

- Thirty ways to do this
- Naturally splits into two sets of 15 (orbits of \( A_7 \))
A Fano Plane
The HS Graph, Again

Vertices

- All 3-sets from a 7-set, 35 “triples”
- One natural half of all Fano planes, 15 “planes”
The HS Graph, Again

Vertices
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Edges
- Edge between a triple and a plane $\iff$ triple is part of the plane
- Edge between a triple and another triple $\iff$ disjoint sets
- Edge between a plane and another plane: never
Independent Sets in HS Graph

- Independent sets (“cocliques”) are vertex sets with no adjacencies
- Maximum independent set in HS has size 15
- HS has 100 such independent sets, all the “same”
Maximal Independent Set
Maximal Independent Set
Maximal Independent Set
Maximal Independent Set
The Odd Graph $O_4$

- Remove an independent set from HS
- Subgraph induced by remaining vertices is $O_4$
  - 35 vertices, the 3-sets from a 7-set
  - Triples are adjacent if disjoint
- Generally $O_m$ is $m$-sets from a $(2m + 1)$-set, with disjoint sets being adjacent
- $O_4$ contains Coxeter graph: degree 3, girth 7, 28 vertices (a $(3,7)$-cage has 24 vertices)
Odd Graph $O_4$
Intersections of Independent Sets

Intersections

- Choose one of the 100 independent sets
- Intersections with the other 99 independent sets are:
  - Empty (7 times)
  - Three vertices (35 times)
  - Five vertices (42 times)
  - Eight vertices (15 times)
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Two Independent Sets with Maximal Intersection

- Two independent sets meeting in 8 vertices
- Symmetric difference has $7 + 7 = 14$ vertices, bipartite
- Induced subgraph is the Heawood Graph, unique $(3, 6)$-cage
- HS graph has 750 copies of Heawood Graph
Heawood Graph (Red-Blue Ind Set, Red-Green Ind Set)
Heawood Graph

altermundus.com
New Graphs with Independent Sets as Vertices

- Construct a new graph
- Vertices: the 100 independent sets of HS
- Edges: join independent sets that are disjoint
- Edges: join independent sets that meet in 8 vertices

This is the Higman-Sims graph

▶ Strongly regular, unique for its parameters
▶ Easiest construction from Steiner system S(3,6,22)
▶ 1 vertex, 22 symbols, 77 blocks stitched together
▶ Splits naturally into two copies of HS (352 ways)
▶ Several other interesting subgraphs
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Automorphisms of the HS Graph

- Permutations of the vertices that take edges to edges
- First, permutations that preserve the 50 edges of the pentagons and pentagrams as a set
- One of order 2, swaps point vertices with line vertices

\[(a, b) \mapsto (3a)x + (2b)\]

\[mx + c \mapsto (m, 2c)\]
Order 2 Permutation, Swaps Pentagons & Pentagrams
Automorphisms of the HS Graph

- More permutations that fix the 50 edges of the pentagons and pentagrams as a set
- These fix edges of pentagons and edges of pentagrams as sets
- Parameters: \(d \neq 0, \ e = \pm 1, \ f, \ g, \ h \in \mathbb{Z}_5\)
  \[(a, b) \mapsto (da + g, fa + eb + h)\]
  \[mx + c \mapsto \frac{f + em}{d}x + \left( ec + h - \frac{f + em}{d}g \right)\]
- \(4 \cdot 2 \cdot 5^3 = 1000\) permutations
Original HS

Hoffman-Singleton Graph
Translation of Pentagons
Original HS
Rotations on Pentagons
Original HS
Automorphism Group of HS Graph

- One order 2 permutation, swaps pentagon and pentagram edges
- 1000 permutations fixing pentagon edges and pentagram edges
- Subgroup of order 2000 fixing pentagon and pentagram edges

The automorphism group is transitive on the 1260 10-packs of pentagons. In other words, there exist vertex permutations taking edges of any 10-pack to the edges of any other 10-pack. Thus, the order of the automorphism group is $2 \cdot 1000 \cdot 1260 = 2,520,000$. 

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- In other words, there exist vertex permutations taking edges of any 10-pack to the edges of any other 10-pack
- Thus, order of automorphism group is

\[ 2 \cdot 1000 \cdot 1260 = 252,000 \]
Movie Two
Miscellaneous Facts About HS

- Strongly regular, unique for its parameters
- Hamiltonian
- Line graph is distance-regular
- Vertex-transitive
- Symmetric (transitive on edges, as ordered pairs of vertices)
- Connections to finite geometries
- Connections to codes
References

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