An Introduction to Algebraic Graph Theory

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Labeling Puzzles

- Assign a single real number value to each circle.
- For each circle, sum the values of adjacent circles.
- Goal:
  Sum at each circle should be a common multiple of the value at the circle.

Common multiple: $-2$
Example Solutions

Common multiple: 3
Example Solutions

Common multiple: 1
Example Solutions

Common multiple: 0
Example Solutions

\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\]

Common multiple: $-1$
Example Solutions

Common multiple: 0
Common multiple: $\frac{1}{2} (r + 1) = \frac{1}{2} (\sqrt{17} + 1)$
A graph is a collection of vertices (nodes, dots) where some pairs are joined by edges (arcs, lines).

The geometry of the vertex placement, or the contours of the edges are irrelevant. The relationships between vertices are important.
Adjacency Matrix

Given a graph, build a matrix of zeros and ones as follows:
Label rows and columns with vertices, in the same order.
Put a 1 in an entry if the corresponding vertices are connected by an edge.
Otherwise put a 0 in the entry.
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```
  u  v  w  x
u  0  1  0  1
v  1  0  1  1
w  0  1  0  1
x  1  1  1  0
```

Always a symmetric matrix with zero diagonal.
Useful for computer representations.

Entrée to linear algebra, especially eigenvalues and eigenvectors.
Symmetry groups of graphs is the other branch of Algebraic Graph Theory.

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An Introduction to Algebraic Graph Theory
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<table>
<thead>
<tr>
<th></th>
<th>u</th>
<th>v</th>
<th>w</th>
<th>x</th>
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<tbody>
<tr>
<td>u</td>
<td>0</td>
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<td>0</td>
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<tr>
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Eigenvalues of Graphs

$\lambda$ is an eigenvalue of a graph

$\iff \lambda$ is an eigenvalue of the adjacency matrix

$\iff A\vec{x} = \lambda\vec{x}$ for some vector $\vec{x}$

Adjacency matrix is real, symmetric

$\Rightarrow$ real eigenvalues, algebraic and geometric multiplicities are equal

minimal polynomial is product of linear factors for distinct eigenvalues

Eigenvalues:

$\lambda = 3 \quad m = 1$

$\lambda = 1 \quad m = 5$

$\lambda = -2 \quad m = 4$
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Eigenvalues of Graphs

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Adjacency matrix is real, symmetric \( \Rightarrow \)
real eigenvalues, algebraic and geometric multiplicities are equal
minimal polynomial is product of linear factors for distinct eigenvalues

![Graph Diagram]

Eigenvalues:

\[ \lambda = 3 \quad m = 1 \]
\[ \lambda = 1 \quad m = 5 \]
\[ \lambda = -2 \quad m = 4 \]
4-Dimensional Cube

- Vertices: Length 4 binary strings
- Join strings differing in exactly one bit
- Generalizes 3-D cube

Eigenvalues:

\[
\begin{align*}
\lambda &= 4 & \lambda &= 2 & \lambda &= 0 & \lambda &= -2 & \lambda &= -4 \\
\lambda &= 2 & \lambda &= 4 & \lambda &= 6 & \lambda &= 4 & \lambda &= 1
\end{align*}
\]
Regular Graphs

A graph is **regular** if every vertex has the same number of edges incident. The **degree** is the common number of incident edges.

**Theorem**

Suppose $G$ is a regular graph of degree $r$. Then

- $r$ is an eigenvalue of $G$
- The multiplicity of $r$ is the number of connected components of $G$

Regular of degree 3 with 2 components implies that $\lambda = 3$ will be an eigenvalue of multiplicity 2.
Proof.

- Let \( \overline{u} \) be the vector where every entry is 1. Then

\[
A\overline{u} = A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} r \\ r \\ \vdots \\ r \end{bmatrix} = r\overline{u}
\]

- For each component of the graph, form a vector with 1’s in entries corresponding to the vertices of the component, and zeros elsewhere.

These eigenvectors form a basis for the eigenspace of \( r \).
(Their sum is the vector \( \overline{u} \) above.)
Labeling Puzzles Explained

The product of a graph’s adjacency matrix with a column vector, $A\vec{u}$, forms sums of entries of $\vec{u}$ for all adjacent vertices.

If $\vec{u}$ is an eigenvector, then these sums should equal a common multiple of the numbers assigned to each vertex. This multiple is the eigenvalue.

So the puzzles earlier were simply asking for eigenvectors (assignments of numbers) and eigenvalues (common multiples) of the adjacency matrix of the graph.
Eigenvalues and Eigenvectors of the Prism

\[ A = \begin{bmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 0
\end{bmatrix} \]

\[ \lambda = 3 \]
\[ \begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix} \]

\[ \lambda = 1 \]
\[ \begin{bmatrix}
1 \\
1 \\
-1 \\
-1 \\
-1 \\
1
\end{bmatrix} , \quad \begin{bmatrix}
0 \\
-1 \\
-1 \\
0 \\
1 \\
1
\end{bmatrix} \]

\[ \lambda = 0 \]
\[ \begin{bmatrix}
0 \\
1 \\
0 \\
1 \\
0 \\
1
\end{bmatrix} , \quad \begin{bmatrix}
1 \\
-1 \\
-1 \\
1 \\
-1 \\
1
\end{bmatrix} \]

\[ \lambda = -2 \]
\[ \begin{bmatrix}
0 \\
-1 \\
-1 \\
1 \\
-1 \\
0
\end{bmatrix} , \quad \begin{bmatrix}
-1 \\
1 \\
-1 \\
0 \\
1 \\
0
\end{bmatrix} \]
Eigenvalues and Eigenvectors

\[ A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \]

\[ \lambda = -1 \quad \lambda = 0 \quad \lambda = \frac{1}{2} \left( \sqrt{17} + 1 \right) \quad \lambda = \frac{1}{2} \left( -\sqrt{17} + 1 \right) \]

\[ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} \sqrt{17} - 1 \\ 4 \\ \sqrt{17} - 1 \\ 4 \end{bmatrix} \quad \begin{bmatrix} \sqrt{17} + 1 \\ -4 \\ \sqrt{17} + 1 \\ -4 \end{bmatrix} \]
Powers of Adjacency Matrices

**Theorem**

Suppose $A$ is the adjacency matrix of a graph. Then the number of walks of length $\ell$ between vertices $v_i$ and $v_j$ is the entry in row $i$, column $j$ of $A^\ell$.

**Proof.**

Base case: $\ell = 1$. Adjacency matrix describes walks of length 1.

$$[A]_{ij} = \sum_{k} [A]_{ik} [A]_{kj} = \sum_{k: v_k \text{ adj } v_j} [A]_{ik} = \text{total ways to walk in } \ell \text{ steps from } v_i \text{ to } v_k, \text{ a neighbor of } v_j.$$
Powers of Adjacency Matrices

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Suppose $A$ is the adjacency matrix of a graph. Then the number of walks of length $\ell$ between vertices $v_i$ and $v_j$ is the entry in row $i$, column $j$ of $A^\ell$.

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Base case: $\ell = 1$. Adjacency matrix describes walks of length 1.

\[
[A^{\ell+1}]_{ij} = [A^\ell A]_{ij} = \sum_{k=1}^{n} [A^\ell]_{ik} [A]_{kj} = \sum_{k: v_k \text{ adj } v_j} [A^\ell]_{ik}
\]

= total ways to walk in $\ell$ steps from $v_i$ to $v_k$, a neighbor of $v_j$
Number of walks of length 3 between vertices 1 and 3?

\[ A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \]

\[ A^3 = \begin{bmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{bmatrix} \]
Characteristic Polynomial

Recall the characteristic polynomial of a square matrix $A$ is $\det(\lambda I - A)$. Roots of this polynomial are the eigenvalues of the matrix.

For the adjacency matrix of a graph on $n$ vertices:

- Coefficient of $\lambda^n$ is 1
- Coefficient of $\lambda^{n-1}$ is zero (trace of adjacency matrix)
- Coefficient of $-\lambda^{n-2}$ is number of edges
- Coefficient of $-\frac{1}{2}\lambda^{n-3}$ is number of triangles

Characteristic polynomial

$\lambda^4 - 5\lambda^2 - 4\lambda + \cdots$
Diameter and Eigenvalues

The diameter of a graph is the longest shortest path. In other words, find the shortest route between each pair of vertices, then ask which pair is farthest apart?

**Theorem**

Suppose $G$ is a graph of diameter $d$. Then $G$ has at least $d + 1$ distinct eigenvalues.

**Proof.**

Minimal polynomial is product of linear factors, one per distinct eigenvalue. There are zero short walks between vertices that are far apart. So the first $d$ powers of $A$ are linearly independent. So $A$ cannot satisfy a polynomial of degree $d$ or less. Thus minimal polynomial has at least $d + 1$ factors hence at least $d + 1$ eigenvalues.
Minimal Polynomial for a Diameter 3 Graph

Distinct Eigenvalues: \( \lambda = 2, 1, -1, -2 \)

Minimal Polynomial:
\[(x - 2)(x - 1)(x - (-1))(x - (-2)) = x^4 - 5x^2 + 4\]

Check that \( A^4 - 5A^2 + 4 = 0 \)
Recent Progress on Diameters and Eigenvalues

Regular graph: \( n \) vertices, degree \( r \), diameter \( d \)
Let \( \lambda \) denote second largest eigenvalue (\( r \) is largest eigenvalue)
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- Alon, Milman, 1985

\[
d \leq 2 \left[ \left( \frac{2r}{r - \lambda} \right)^{\frac{1}{2}} \log_2 n \right]
\]
Recent Progress on Diameters and Eigenvalues

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  \[
  d \leq 2 \left[ \left( \frac{2r}{r - \lambda} \right)^{\frac{1}{2}} \log_2 n \right]
  \]

- Mohar, 1991
  \[
  d \leq 2 \left[ \left( \frac{2r - \lambda}{4(r - \lambda)} \right) \ln (n - 1) \right]
  \]
Bipartite Graphs

A graph is bipartite if the vertex set can be split into two parts, so that every edge goes from part to the other. Identical to being “two-colorable.”

Theorem

A graph is bipartite if and only if there are no cycles of odd length.
Eigenvalues of Bipartite Graphs

**Theorem**

Suppose $G$ is a bipartite graph with eigenvalue $\lambda$. Then $-\lambda$ is also an eigenvalue of $G$.

\[
\begin{array}{c|c|c|c|c}
\lambda = 3 & \lambda = -3 & \lambda = 1 & \lambda = -1 \\
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
\end{bmatrix} & \begin{bmatrix}
1 \\
-1 \\
1 \\
-1 \\
\end{bmatrix} & \begin{bmatrix}
1 \\
0 \\
-1 \\
0 \\
\end{bmatrix} & \begin{bmatrix}
1 \\
0 \\
1 \\
0 \\
\end{bmatrix}
\end{array}
\]
Proof.

Order the vertices according to the two parts. Then

\[
A = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix}
\]

For an eigenvector \( \vec{u} = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \) of \( A \) for \( \lambda \),

\[
\begin{bmatrix} \lambda \vec{u}_1 \\ \lambda \vec{u}_2 \end{bmatrix} = \lambda \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = \lambda \vec{u} = A\vec{u} = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = \begin{bmatrix} B\vec{u}_2 \\ B^t \vec{u}_1 \end{bmatrix}
\]

Now set \( \vec{v} = \begin{bmatrix} -\vec{u}_1 \\ \vec{u}_2 \end{bmatrix} \) and compute,

\[
A\vec{v} = \begin{bmatrix} 0 & B \\ B^t & 0 \end{bmatrix} \begin{bmatrix} -\vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = \begin{bmatrix} B\vec{u}_2 \\ -B^t \vec{u}_1 \end{bmatrix} = \begin{bmatrix} \lambda \vec{u}_1 \\ -\lambda \vec{u}_2 \end{bmatrix} = -\lambda \begin{bmatrix} -\vec{u}_1 \\ \vec{u}_2 \end{bmatrix} = -\lambda \vec{v}
\]
Bipartite Graphs on an Odd Number of Vertices

Theorem

Suppose $G$ is a bipartite graph with an odd number of vertices. Then zero is an eigenvalue of the adjacency matrix of $G$.

Proof.

Pair each eigenvalue with its negative. At least one eigenvalue must then equal its negative.
Application: Friendship Theorem

**Theorem**

*Suppose that at a party every pair of guests has exactly one friend in common. Then there are an odd number of guests, and one guest knows everybody.*

- Model party with a graph
- Vertices are guests, edges are friends
- Two possibilities:
Application: Friendship Theorem

Theorem

Suppose that at a party every pair of guests has exactly one friend in common. Then there are an odd number of guests, and one guest knows everybody.

- Model party with a graph
- Vertices are guests, edges are friends
- Two possibilities:
  1. Graph to the right
  2. Algebraic graph theory say the adjacency matrix has eigenvalues with non-integer multiplicities
Application: Buckminsterfullerene

- Molecule composed solely of 60 carbon atoms
- “Carbon-60,” “buckyball”
- At each atom two single bonds and one double bond describe a graph that is regular of degree 3
- Graph is the skeleton of a truncated icosahedron, a soccer ball
- $60 \times 60$ adjacency matrix
- Eigenvalues of adjacency matrix predict peaks in mass spectrometry
- First created in 1985 (1996 Nobel Prize in Chemistry)
Moral of the Story

- Graph properties predict linear algebra properties
  (regular of degree $r \implies r$ is an eigenvalue)
Moral of the Story

- Graph properties predict linear algebra properties
  (regular of degree $r \Rightarrow r$ is an eigenvalue)
- Linear algebra properties predict graph properties
  (functions of two largest eigenvalues bound the diameter)
Moral of the Story

- Graph properties predict linear algebra properties
  (regular of degree $r \Rightarrow r$ is an eigenvalue)
- Linear algebra properties predict graph properties
  (functions of two largest eigenvalues bound the diameter)
- Do eigenvalues characterize graphs? No.
Bibliography

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- Chris Godsil, Gordon Royle, *Algebraic Graph Theory*
  Springer, 2001

- Robert Beezer, *A First Course in Linear Algebra*
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