Counting Subgraphs in Regular Graphs

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Problem Statement

For a regular graph on n vertices, of degree r, determine the number of matchings with m edges.

- A matching is a subgraph of disjoint edges.

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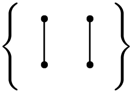
- Regularity is key!
- Notation:

 is the number of subgraphs that are 2-matchings.

1-Matchings

EZ

$\Bigg\{$  $\Bigg\} = \frac{nr}{2}$

- Depends only on n and r.
- Independent of the particular graph.

2-Matchings

Choose a vertex, choose two incident edges:

{  } $= n\binom{r-1}{2}$

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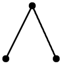
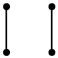
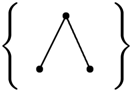
Total of all 2-edge subgraphs:

$\binom{\frac{nr}{2}}{2} = \Big\{$  $\Big\} + \Big\{$  $\Big\}$

2-Matchings

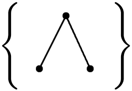
Choose a vertex, choose two incident edges:



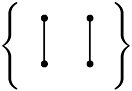
$$= n\binom{r-1}{2}$$

Total of all 2-edge subgraphs:

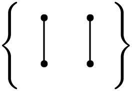
$$\binom{\frac{nr}{2}}{2} =$$



$$+$$



Solve:

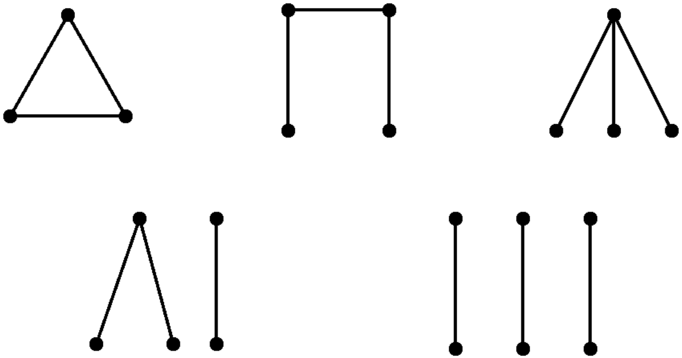


$$= \frac{1}{8} n\, r\, (nr - 4r + 2)$$

- Depends only on n and r.
- Independent of the particular graph.

3-Matchings

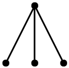
Five possible subgraphs on three edges. How many of each?



We are after the number of 3-matchings. Eventually.

3-Matchings, Part I

Choose a vertex, choose three incident edges:

$\{$  $\} = n\binom{r-1}{3}$

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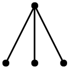
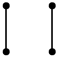
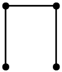
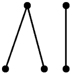
$\left\{\right.$  $\left.\right\} = n\binom{r-1}{3}$

Start with a 2-matching, choose one of 4 vertices, add one of $r-1$ incident edges. Builds paths of length 3, and subgraphs with a path of length 2 and a disjoint edge.

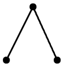
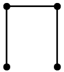
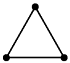
Double-counts each of these though!

$4(r-1)\left\{\right.$  $\left.\right\} = 2\left\{\right.$  $\left.\right\} + 2\left\{\right.$  $\left.\right\}$

3-Matchings, Part II

Start with a path of length 2, choose one of 2 end vertices, add one of $r-1$ incident edges. Builds paths of length 3, and triangles.

Double-counts paths, overcounts triangles by a factor of 6.

$2(r-1)$ {  } $= 2$ {  } $+ 6$ {  }

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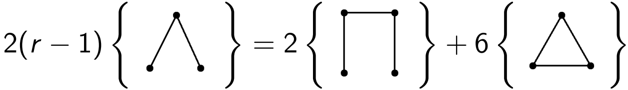
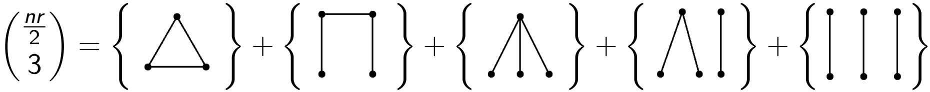
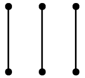
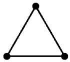


Sum all subgraphs on 3 edges:



3-Matchings, Solution

Solve 4 linear equations in 5 unknowns:

$\Big\{$  $\Big\} = \frac{1}{48} nr \left(n^2 r^2 - 12nr^2 + 40r^2 + 6nr - 48r + 16\right) - \Big\{$  $\Big\}$

- Depends on n and r **and** the number of triangles.
- Independent of the particular graph.

General Approach

Can't keep this up. Need a systematic approach.

- Begin with a subgraph with m edges and a vertex of degree 1.

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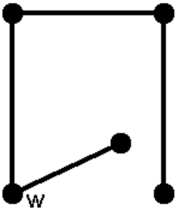
- Begin with a subgraph with m edges and a vertex of degree 1.
- Remove edge incident to degree 1 vertex. Call other endpoint w.
- Identify vertices "isomorphic" to w.
- Add back a single edge, attaching one end at vertices like w.
- Determine the types of subgraphs formed.
- Determine the amount of overcounting.

Example, 4 Edges

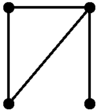
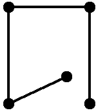
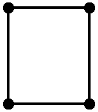
Begin with a path having 4 edges.
Remove an edge incident to a vertex of degree 1. Label other endpoint w.
In the path on 3 edges that remains, there is one other vertex like w.



Add back an edge at w, considering all vertices as possibilities for the other end of the new edge.

Example, 4 Edges

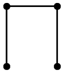
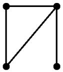
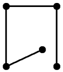
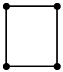
What subgraphs result? How many of each? Overcounting factor?

Type		Overcount
Triangle w/Pendant		2x
Path		2x
Circuit		8x

Example, 4 Edges

2 vertices like w.
$r - 1$ ways to attach back an edge.

Counting a set of subgraphs (each with a labeled vertex at w) in two different ways yields:

$2(r-1)\Big\{$  $\Big\} = 2\Big\{$  $\Big\} + 2\Big\{$  $\Big\} + 8\Big\{$  $\Big\}$

System of Linear Equations

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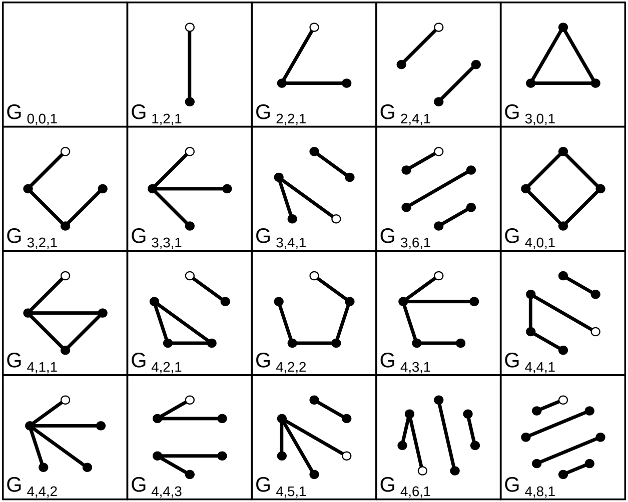
- Subgraphs with degree 1 vertices are dependent variables.
- Order subgraph types on edges, then number of degree 1 vertices.
- System has lower-triangular coefficient matrix, nearly homogeneous.

4-Matchings

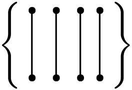
All subgraphs on 4 edges or less. *w* is adjacent to open-circle vertex.



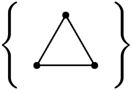
$$
\begin{aligned}
g_{0,0,1} &= 1 \\
n\,r\,g_{0,0,1} &= 2\,g_{1,2,1} \\
2(r-1)\,g_{1,2,1} &= 2\,g_{2,2,1} \\
(n-2)\,r\,g_{1,2,1} &= 2\,g_{2,2,1} + 4\,g_{2,4,1} \\
2(r-1)\,g_{2,2,1} &= 6\,g_{3,0,1} + 2\,g_{3,2,1} \\
1(r-2)\,g_{2,2,1} &= 3\,g_{3,3,1} \\
4(r-1)\,g_{2,4,1} &= 2\,g_{3,2,1} + 2\,g_{3,4,1} \\
(n-4)\,r\,g_{2,4,1} &= 2\,g_{3,4,1} + 6\,g_{3,6,1} \\
3(r-2)\,g_{3,0,1} &= 1\,g_{4,1,1} \\
(n-3)\,r\,g_{3,0,1} &= 1\,g_{4,1,1} + 2\,g_{4,2,1} \\
2(r-1)\,g_{3,2,1} &= 8\,g_{4,0,1} + 2\,g_{4,1,1} + 2\,g_{4,2,2} \\
2(r-2)\,g_{3,2,1} &= 2\,g_{4,1,1} + 2\,g_{4,3,1} \\
2(r-1)\,g_{3,4,1} &= 6\,g_{4,2,1} + 2\,g_{4,2,2} + 2\,g_{4,4,1} \\
1(r-3)\,g_{3,3,1} &= 4\,g_{4,4,2} \\
2(r-1)\,g_{3,4,1} &= 2\,g_{4,2,2} + 1\,g_{4,3,1} + 4\,g_{4,4,3} \\
1(r-2)\,g_{3,4,1} &= 1\,g_{4,3,1} + 3\,g_{4,5,1} \\
6(r-1)\,g_{3,6,1} &= 2\,g_{4,4,1} + 2\,g_{4,6,1} \\
(n-6)\,r\,g_{3,6,1} &= 2\,g_{4,6,1} + 8\,g_{4,8,1}
\end{aligned}
$$

$$g_{0,0,1} = 1$$
$$g_{1,2,1} = \frac{nr}{2}$$
$$g_{2,2,1} = \frac{n\,(-1+r)\,r}{2}$$
$$g_{2,4,1} = \frac{nr\,(2-4r+nr)}{8}$$
$$g_{3,2,1} = \frac{n(-1+r)^2 r}{2} - 3g_{3,0,1}$$
$$g_{3,3,1} = \frac{n\,(-2+r)\,(-1+r)\,r}{6}$$
$$g_{3,4,1} = \frac{n\,(-1+r)\,r\,(4-6r+nr)}{4} + 3g_{3,0,1}$$
$$g_{3,6,1} = \frac{nr\,\left(16-48r+6nr+40r^2-12nr^2+n^2r^2\right)}{48} - g_{3,0,1}$$
$$g_{4,1,1} = (-6+3r)\,g_{3,0,1}$$
$$g_{4,2,1} = \left(3-3r+\frac{nr}{2}\right)g_{3,0,1}$$
$$g_{4,2,2} = \frac{n(-1+r)^3 r}{2} + (9-6r)\,g_{3,0,1} - 4g_{4,0,1}$$
$$g_{4,3,1} = \frac{n\,(-2+r)\,(-1+r)^2 r}{2} + (12-6r)\,g_{3,0,1}$$
$$g_{4,4,1} = \frac{n(-1+r)^2 r\,(6-8r+nr)}{4} + \left(-21+18r-\frac{3nr}{2}\right)g_{3,0,1} + 4g_{4,0,1}$$
$$g_{4,4,2} = \frac{n\,(-3+r)\,(-2+r)\,(-1+r)\,r}{24}$$
$$g_{4,4,3} = \frac{n(-1+r)^2 r\,(8-9r+nr)}{8} + (-9+6r)\,g_{3,0,1} + 2g_{4,0,1}$$
$$g_{4,5,1} = \frac{n\,(-2+r)\,(-1+r)\,r\,(6-8r+nr)}{12} + (-6+3r)\,g_{3,0,1}$$
$$g_{4,6,1} = \frac{n\,(-1+r)\,r\,\left(40-104r+10nr+72r^2-16nr^2+n^2r^2\right)}{16} + \left(24-21r+\frac{3nr}{2}\right)g_{3,0,1} - 4g_{4,0,1}$$
$$g_{4,8,1} = \frac{nr}{384}\left(240-960r+76nr+1344r^2-240nr^2+12n^2r^2-672r^3+208nr^3-24n^2r^3+n^3r^3\right) + \left(-6+6r-\frac{nr}{2}\right)g_{3,0,1} + g_{4,0,1}$$

4-Matchings Solution



$$= \frac{nr}{384}\left(240 - 960r + 76nr + 1344r^2 - 240nr^2 + 12n^2r^2 - 672r^3 + 208nr^3 - 24n^2r^3 + n^3r^3\right) + \left(-6 + 6r - \frac{nr}{2}\right)$$



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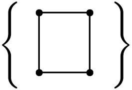

- Applys to any regular graph.
- Depends on n, r, and the number of triangles and squares.

Designs

The pair $(V, \mathcal{B})$ is a t-(v, k, λ) *design* if V is a set of v elements called points (or vertices) and $\mathcal{B}$ is a set of k element subsets of V called blocks (or lines) with the property that every t-element subset of V is a subset of exactly λ blocks from $\mathcal{B}$.

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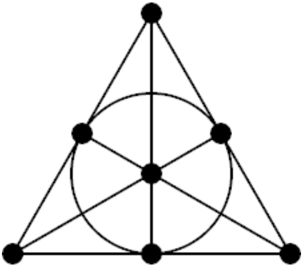

Fano Plane
Projective Plane of Order 2
Steiner Triple System
2-$(7, 3, 1)$ Design

Blocks:
123 345 567 257 147 367 246

Generalize to Designs

- Horak, et al; results for Steiner Triple Systems, 2-$(v, 3, 1)$ designs.

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- "Free" variables are configuration counts for configurations where every block has more than t points that occur in two or more blocks of the configuration.
- For graphs, this is "every edge has 2 vertices of degree 2 or more." i.e. no vertices of degree 1.

Applications

- Graphs of high girth lack small cycles.
- Small subgraphs are acyclic.
- “Free” variables are all zero.
- Counts for small subgraphs are determined just by n and r.

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- Existence of designs?
- Conclude that configuration counts are negative or fractional?

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