Counting Subgraphs in Regular Graphs

Rob Beezer
beezer@ups.edu

Department of Mathematics and Computer Science
University of Puget Sound

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Problem Statement

For a regular graph on \( n \) vertices, of degree \( r \), determine the number of matchings with \( m \) edges.

- A matching is a subgraph of disjoint edges.
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For a regular graph on $n$ vertices, of degree $r$, determine the number of matchings with $m$ edges.

- A matching is a subgraph of disjoint edges.
- Regularity is key!
- Notation:

\[
\left\{ \begin{array}{c}
  \\
  \\
\end{array} \right. 
\] is the number of subgraphs that are 2-matchings.
1-Matchings

EZ

\[ \{ \quad \} = \frac{nr}{2} \]

- Depends only on \( n \) and \( r \).
- Independent of the particular graph.
2-Matchings

Choose a vertex, choose two incident edges:

\[ \{ \begin{array}{ccc}
& * & \\
& \downarrow & \\
& * & \\
\end{array} \} = n \binom{r - 1}{2} \]
2-Matchings

Choose a vertex, choose two incident edges:

\[ \left\{ \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{vertex-edge}}
\end{array}
\end{array} \right\} = n \binom{r-1}{2} \]

Total of all 2-edge subgraphs:

\[ \binom{\frac{nr}{2}}{2} = \left\{ \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{vertex-edge}}
\end{array}
\end{array} \right\} + \left\{ \begin{array}{c}
\begin{array}{c}
\text{\includegraphics{isolated-labeled}}
\end{array}
\end{array} \right\} \]
2-Matchings

Choose a vertex, choose two incident edges:

\[
\{ \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array} \} = n \binom{r - 1}{2}
\]

Total of all 2-edge subgraphs:

\[
\binom{\frac{nr}{2}}{2} = \{ \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array} \} + \{ \begin{array}{c}
\bigcirc \\
\bigcirc
\end{array} \}
\]

Solve:

\[
\{ \begin{array}{c}
\bigcirc \\
\bigcirc \\
\bigcirc
\end{array} \} = \frac{1}{8} nr (nr - 4r + 2)
\]

- Depends only on \( n \) and \( r \).
- Independent of the particular graph.
3-Matchings

Five possible subgraphs on three edges. How many of each?

We are after the number of 3-matchings. Eventually.
3-Matchings, Part I

Choose a vertex, choose three incident edges:

\[ \binom{r-1}{3} = n \binom{r-1}{3} \]
3-Matchings, Part I

Choose a vertex, choose three incident edges:

\[
\binom{r-1}{3} = n \binom{r-1}{3}
\]

Start with a 2-matching, choose one of 4 vertices, add one of \( r - 1 \) incident edges. Builds paths of length 3, and subgraphs with a path of length 2 and a disjoint edge.

Double-counts each of these though!

\[
4(r - 1) \binom{r-1}{3} = 2 \binom{r-1}{3} + 2 \binom{r-1}{3}
\]
3-Matchings, Part II

Start with a path of length 2, choose one of 2 end vertices, add one of \( r - 1 \) incident edges. Builds paths of length 3, and triangles.

Double-counts paths, overcounts triangles by a factor of 6.

\[
2(r - 1) \left\{ \begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet 
\end{array} \right\} = 2 \left\{ \begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet 
\end{array} \right\} + 6 \left\{ \begin{array}{c}
  \bullet \\
  \bullet \\
  \bullet 
\end{array} \right\}
\]
3-Matchings, Part II

Start with a path of length 2, choose one of 2 end vertices, add one of $r - 1$ incident edges. Builds paths of length 3, and triangles.

Double-counts paths, overcounts triangles by a factor of 6.

$$2(r - 1) \{ \text{Path of length 3} \} = 2 \{ \text{Path of length 2} \} + 6 \{ \text{Triangle} \}$$

Sum all subgraphs on 3 edges:

$$\binom{nr}{3} = \{ \text{Path of length 3} \} + \{ \text{Path of length 2} \} + \{ \text{Path of length 1} \} + \{ \text{Triangle} \} + \{ \text{Complete graph} \}$$
3-Matchings, Solution

Solve 4 linear equations in 5 unknowns:

\[
\left\{ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \right\} = \frac{1}{48} nr \left( n^2 r^2 - 12nr^2 + 40r^2 + 6nr - 48r + 16 \right) - \left\{ \begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array} \right\}
\]

- Depends on \( n \) and \( r \) and the number of triangles.
- Independent of the particular graph.
General Approach

Can’t keep this up. Need a systematic approach.

- Begin with a subgraph with $m$ edges and a vertex of degree 1.
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- Identify vertices “isomorphic” to $w$. 
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- Add back a single edge, attaching one end at vertices like \( w \).
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- Begin with a subgraph with $m$ edges and a vertex of degree 1.
- Remove edge incident to degree 1 vertex. Call other endpoint $w$.
- Identify vertices “isomorphic” to $w$.
- Add back a single edge, attaching one end at vertices like $w$.
- Determine the types of subgraphs formed.
- Determine the amount of overcounting.
Example, 4 Edges

Begin with a path having 4 edges. Remove an edge incident to a vertex of degree 1. Label other endpoint $w$. In the path on 3 edges that remains, there is one other vertex like $w$.

Add back an edge at $w$, considering all vertices as possibilities for the other end of the new edge.
**Example, 4 Edges**

What subgraphs result? How many of each? Overcounting factor?

<table>
<thead>
<tr>
<th>Type</th>
<th>Overcount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle w/Pendant</td>
<td>2x</td>
</tr>
<tr>
<td>Path</td>
<td>2x</td>
</tr>
<tr>
<td>Circuit</td>
<td>8x</td>
</tr>
</tbody>
</table>

![Diagram of subgraphs]
Example, 4 Edges

2 vertices like $w$.

$r - 1$ ways to attach back an edge.

Counting a set of subgraphs (each with a labeled vertex at $w$) in two different ways yields:

\[2(r - 1) \{ \begin{array}{c} \text{\includegraphics{example_graph_1.png}} \end{array} \} = 2 \{ \begin{array}{c} \text{\includegraphics{example_graph_2.png}} \end{array} \} + 2 \{ \begin{array}{c} \text{\includegraphics{example_graph_3.png}} \end{array} \} + 8 \{ \begin{array}{c} \text{\includegraphics{example_graph_4.png}} \end{array} \} \]
System of Linear Equations

- Create a system of linear equations in subgraph counts.
- Coefficients are constants, functions of $n$ and $r$. 
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- Subgraphs with degree 1 vertices are dependent variables.
System of Linear Equations

- Create a system of linear equations in subgraph counts.
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- Subgraphs with no degree 1 vertices are “free” variables.
- Subgraphs with degree 1 vertices are dependent variables.
- Order subgraph types on edges, then number of degree 1 vertices.
- System has lower-triangular coefficient matrix, nearly homogeneous.
4-Matchings

All subgraphs on 4 edges or less. \( w \) is adjacent to open-circle vertex.
\begin{align*}
g_{0,0,1} &= 1 \\
nr g_{0,0,1} &= 2g_{1,2,1} \\
2(r - 1)g_{1,2,1} &= 2g_{2,2,1} \\
(n - 2)r g_{1,2,1} &= 2g_{2,2,1} + 4g_{2,4,1} \\
2(r - 1)g_{2,2,1} &= 6g_{3,0,1} + 2g_{3,2,1} \\
1(r - 2)g_{2,2,1} &= 3g_{3,3,1} \\
4(r - 1)g_{2,4,1} &= 2g_{3,2,1} + 2g_{3,4,1} \\
(n - 4)r g_{2,4,1} &= 2g_{3,4,1} + 6g_{3,6,1} \\
3(r - 2)g_{3,0,1} &= 1g_{4,1,1} \\
(n - 3)r g_{3,0,1} &= 1g_{4,1,1} + 2g_{4,2,1} \\
2(r - 1)g_{3,2,1} &= 8g_{4,0,1} + 2g_{4,1,1} + 2g_{4,2,2} \\
2(r - 2)g_{3,2,1} &= 2g_{4,1,1} + 2g_{4,3,1} \\
2(r - 1)g_{3,4,1} &= 6g_{4,2,1} + 2g_{4,2,2} + 2g_{4,4,1} \\
1(r - 3)g_{3,3,1} &= 4g_{4,4,2} \\
2(r - 1)g_{3,4,1} &= 2g_{4,2,2} + 1g_{4,3,1} + 4g_{4,4,3} \\
1(r - 2)g_{3,4,1} &= 1g_{4,3,1} + 3g_{4,5,1} \\
6(r - 1)g_{3,6,1} &= 2g_{4,4,1} + 2g_{4,6,1} \\
(n - 6)r g_{3,6,1} &= 2g_{4,6,1} + 8g_{4,8,1}
\end{align*}
\[g_{0,0,1} = 1\]
\[g_{1,2,1} = \frac{nr}{2}\]
\[g_{2,2,1} = \frac{n(-1 + r) r}{2}\]
\[g_{2,4,1} = \frac{nr(2 - 4r + nr)}{8}\]
\[g_{3,2,1} = \frac{n(-1 + r)^2r}{2} - 3g_{3,0,1}\]
\[g_{3,3,1} = \frac{n(-2 + r)(-1 + r) r}{6}\]
\[g_{3,4,1} = \frac{n(-1 + r) r (4 - 6r + nr)}{4} + 3g_{3,0,1}\]
\[g_{3,6,1} = \frac{nr (16 - 48r + 6nr + 40r^2 - 12nr^2 + n^2r^2)}{48} - g_{3,0,1}\]
\[g_{4,1,1} = (-6 + 3r) g_{3,0,1}\]
\[g_{4,2,1} = \left(3 - 3r + \frac{nr}{2}\right) g_{3,0,1}\]
\[g_{4,2,2} = \frac{n(-1 + r)^3r}{2} + (9 - 6r) g_{3,0,1} - 4g_{4,0,1}\]
\[g_{4,3,1} = \frac{n(-2 + r)(-1 + r)^2r}{2} + (12 - 6r) g_{3,0,1}\]
\[g_{4,4,1} = \frac{n(-1 + r)^2r (6 - 8r + nr)}{4} + \left(-21 + 18r - \frac{3nr}{2}\right) g_{3,0,1} + 4g_{4,0,1}\]
\[g_{4,4,2} = \frac{n(-3 + r)(-2 + r)(-1 + r) r}{24}\]
\[g_{4,4,3} = \frac{n(-1 + r)^2r (8 - 9r + nr)}{8} + (-9 + 6r) g_{3,0,1} + 2g_{4,0,1}\]
\[g_{4,5,1} = \frac{n(-2 + r)(-1 + r)r (6 - 8r + nr)}{12} + (-6 + 3r) g_{3,0,1}\]
\[g_{4,6,1} = \frac{n(-1 + r)r (40 - 104r + 10nr + 72r^2 - 16nr^2 + n^2r^2)}{16} + \left(24 - 21r + \frac{3nr}{2}\right) g_{3,0,1} - 4g_{4,0,1}\]
\[g_{4,8,1} = \frac{nr}{384} \left(240 - 960r + 76nr + 1344r^2 - 240n^2r^2 - 672r^3 + 208nr^3 - 24n^2r^3 + n^3r^3\right) + \left(-6 + 6r - \frac{nr}{2}\right) g_{3,0,1} + g_{4,0,1}\]
4-Matchings Solution

\[
\{\text{\textbullet\textbullet\textbullet\textbullet}\} = \frac{nr}{384} \left(240 - 960r + 76nr + 1344r^2 - 240nr^2 + 12n^2r^2 - 672r^3 + 208nr^3 - 24n^2r^3 + n^3r^3 \right) + \\
\left(-6 + 6r - \frac{nr}{2}\right) \{\text{\textbullet\textbullet\textbullet}\} + \{\text{\textbullet\textbullet\textbullet\textbullet}\}
\]

- Applies to any regular graph.
- Depends on \(n\), \(r\), and the number of triangles and squares.
The pair \((V, B)\) is a \(t-(v, k, \lambda)\) design if \(V\) is a set of \(v\) elements called points (or vertices) and \(B\) is a set of \(k\) element subsets of \(V\) called blocks (or lines) with the property that every \(t\)-element subset of \(V\) is a subset of exactly \(\lambda\) blocks from \(B\).
Designs

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![Diagram of a Fano Plane]

Fano Plane
Projective Plane of Order 2
Steiner Triple System
2-(7, 3, 1) Design

Blocks:
123  345  567  257  147  367  246
Generalize to Designs

- Horak, et al; results for Steiner Triple Systems, 2-(v, 3, 1) designs.
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- In a design, subgraphs are just subsets of blocks, “configurations.”
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- Same types of linear equations.
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- Same types of linear equations.
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- “Free” variables are configuration counts for configurations where every block has more than $t$ points that occur in two or more blocks of the configuration.
- For graphs, this is “every edge has 2 vertices of degree 2 or more.” i.e. no vertices of degree 1.
Applications

- Graphs of high girth lack small cycles.
- Small subgraphs are acyclic.
- “Free” variables are all zero.
- Counts for small subgraphs are determined just by $n$ and $r$. 
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- Small subgraphs are acyclic.
- “Free” variables are all zero.
- Counts for small subgraphs are determined just by $n$ and $r$.
- Existence of designs?
- Conclude that configuration counts are negative or fractional?
Bibliography

- R.A. Beezer, The number of subgraphs of a regular graph
  *Congressus Numerantium*, 100:89–96, 1994

- P. Horak, N.K.C. Phillips, W.D. Wallis, and J.L. Yucas
  Counting frequencies of configurations in Steiner triple systems

- M.J. Grannell, T. S. Griggs, Configurations in Steiner triple systems
  Research Notes in Math. 403, 103–126, 1999

- R.A. Beezer, Counting configurations in designs