Math 390 Monday, April 19 Determinants

$$P_A(x) = \det(xI - A) \quad (290)$$

$$= \prod_{i=1}^{n} (x - \lambda_i) \quad (390)$$

w/ repeats

(-1) for each row

$$P_A(0) = \det(0I - A) = \det(-A) = (-1)^n \det(A)$$

$$P_A(0) = \prod_{i=1}^{n} (0 - \lambda_i) = (-1)^n \prod_{i=1}^{n} \lambda_i$$

<u>Thm</u> $\det(A) = \prod_{i=1}^{n} \lambda_i$

This could be the <u>definition</u> of the determinant.

<u>Defn</u> A square matrix, the <u>trace</u> of A, is the sum of the diagonal entries, $tr(A) = \sum_{i=1}^{n} [A]_{ii}$

Given A, find T that is upper-triangular & similar to A.
Then SUT, or UTEC, says the diagonal entries are eigenvalues.

$$\operatorname{tr}(T) = \sum_{i=1}^{n} \lambda_i$$

$$P_A(x) = \prod_{i=1}^{n} (x-\lambda_i) = (x-\lambda_1)(x-\lambda_2) \cdots (x-\lambda_n)$$
$$= x^n + (-\lambda_1 \cdots -\lambda_n)x^{n-1} + \cdots$$
$$= x^n - \operatorname{tr}(T)x^{n-1}$$

Any matrix similar to T (notably A) will have the same characteristic polynomial, so the trace is always the negative of the sum of the eigenvalues.

<u>Fact</u> $\det(A)$ & $\operatorname{tr}(A)$ are properties of a linear transformation, since they are the same for any matrix representation

Permutation Definition of a Determinant

Defn A <u>permutation</u> is a 1-1 & onto function from a finite set to itself.

Ex $X = \{1, 2, 3\}$

$$f: \begin{array}{l} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{array} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{array}{l} \leftarrow \text{domain} \\ \leftarrow \text{codomain} \end{array} \qquad 2\,3\,1$$

rearrangement

"natural" order

Ex $X = \{a, b, c, d\}$

Permutations: $b\,a\,d\,c$

$c\,b\,a\,d$ "fixes" b & d

$a\,b\,c\,d$ identity permutation

$a\ b\ c\ d$

Inverse: $b\,d\,a\,c$ composition is the identity

has inverse $c\,a\,d\,b$

function

Defn Suppose $A = (a_{ij})$ is an $n \times n$ matrix. S_n set of all permutations of $\{1, 2, \ldots, n\}$. Then

$$\det(A) = \sum_{\sigma \in S_n} \underline{\mathrm{sign}(\sigma)}\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

$+1$ or -1

Ex

$$A = \begin{bmatrix} 2 & 1 & \boxed{2} \\ \boxed{-1} & 3 & -2 \\ -1 & \boxed{2} & 1 \end{bmatrix}$$

$\sigma = 3\,1\,2 \qquad \begin{pmatrix} 1 \to 3 \\ 2 \to 1 \\ 3 \to 2 \end{pmatrix}$

$a_{1\sigma(1)}\, a_{2\sigma(2)}\, a_{3\sigma(3)}$

$a_{13} \quad a_{21} \quad a_{32}$

$= 2(-1)(2) = -4$