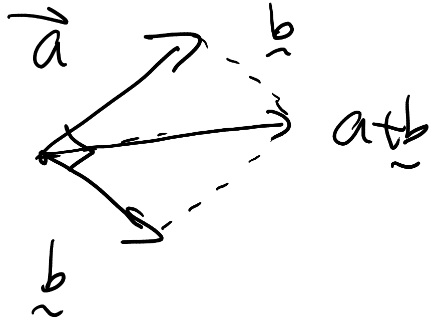
Math 390 Friday, April 16 Least Squares
Characteristic Polynomial

General Pythagorean Theorem

a, b orthogonal $\Rightarrow \|a+b\|^2 = \|a\|^2 + \|b\|^2$



Proof $\|a+b\|^2 = \langle a+b, a+b\rangle$

$$= \langle a, a\rangle + \cancelto{0}{\langle a, b\rangle} + \cancelto{0}{\langle b, a\rangle} + \langle b, b\rangle$$

$$= \|\vec{a}\|^2 + \|b\|^2$$

Theorem 4.1.17 (SCLA)

$r(x) = \|Ax - b\|$ is minimized by $\hat{x}$, solution to $A^*A\hat{x} = A^*b$

least squares

OR $\min\limits_{x} \|Ax - b\| = \|\underbrace{A\hat{x}}_{\hat{b}} - b\| = \|r\|$

residual

<u>Proof</u>

$$A\underset{\sim}{x} - \underset{\sim}{b} = A\underset{\sim}{x} - A\hat{\underset{\sim}{x}} + A\hat{\underset{\sim}{x}} - \underset{\sim}{b}$$

$$= \underbrace{A(\underset{\sim}{x} - \hat{\underset{\sim}{x}})}_{\text{in } C(A)} + \underbrace{A\hat{\underset{\sim}{x}} - \underset{\sim}{b}}_{\text{in } N(A^*)} \quad (\hat{\underset{\sim}{b}} - \underset{\sim}{b} \text{ orthogonal})$$

Corollary 1.3.8 $\Rightarrow$ $\mathbb{C}^m = C(A) \oplus C(A)^\perp = C(A) \oplus N(A^*)$

So $\|A\underset{\sim}{x} - b\|^2 = \|A(\underset{\sim}{x} - \hat{\underset{\sim}{x}}) + A\hat{\underset{\sim}{x}} - b\|^2$

$$= \|A\underset{\sim}{x} - \hat{\underset{\sim}{x}}\|^2 + \|A\hat{\underset{\sim}{x}} - \underset{\sim}{b}\|^2 \quad \text{(Pythagorean Theorem)}$$

This is always bigger than this

$$\geq \|A\hat{\underset{\sim}{x}} - \underset{\sim}{b}\|^2$$

max / <u>min</u>

$$r(a,b) = \left\| \begin{bmatrix} \vdots & c_1 \\ \vdots & c_2 \\ \vdots & \vdots \\ \vdots & c_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \right\|^2$$

Find the values of a & b that minimize this, sum of squares: $\frac{\partial r}{\partial a} = 0$, $\frac{\partial r}{\partial b} = 0$

Determinants

~~Three~~ Four ways to define a determinant. You've seen one.

Characteristic Polynomial

290: $p_A(x) = \det(A - xI)$

leading coefficient $(-1)^n$

"Late Determinants" in 290: $p_A(x) = (x-\lambda_1)^{\alpha_A(\lambda_1)}(x-\lambda_2)^{\alpha_A(\lambda_2)}\cdots(x-\lambda_k)^{\alpha_A(\lambda_k)}$

leading coefficient 1

Definitions

(Section CP)

Theorem (CPD) $p_A(x) = \det(xI - A)$

using this definition

$(-1)^n \det(A - xI) = \det(xI - A)$

Proof

A similar to

$$\begin{bmatrix} \lambda_1 & x & x & x & x \\ & \lambda_2 & & x & x \\ & & \lambda_3 & x & x \\ & 0 & & \ddots & \\ & & & & \lambda_n \end{bmatrix}$$

Theorem UTEC

$$\det(xI - A) = \det \begin{bmatrix} x-\lambda_1 & & & \times \\ & x-\lambda_2 & & \times \\ & & \ddots & \\ 0 & & & x-\lambda_n \end{bmatrix} = (x-\lambda_1)(x-\lambda_2)\cdots(x-\lambda_n)$$

$$= (x-\lambda_1)^{\alpha_A(\lambda_1)} \cdots\cdots (x-\lambda_k)^{\alpha_A(\lambda_k)}$$