Math 390 Thursday, April 15 Orthogonal Projectors, Least Squares

Projectors: $P^2 = P$

Decomposition: $\mathbb{C}^n = C(P) \oplus N(P)$

Orthogonal Projectors

Defn A projector P is orthogonal if $N(P) = C(P)^\perp$

orthogonal complement

Know from before

$$\mathbb{C}^n = C(A) \oplus C(A)^\perp$$
$$= C(A) \oplus N(A^*)$$

projectors $\Rightarrow$ $= C(P) \oplus N(P)$ $\Rightarrow P = P^*$?

Friday - Determinants

Review FCLA

Sunday - Draft Project

(*) Feedback

Next Sunday

(*) Grade

New SCLA posted

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New FCLA this afternoon

Last section CP

Theorem P projector.

P <u>orthogonal</u> projector $\iff$ P Hermitian

Proof Assume P Hermitian. Show $N(P) = C(P)^{\perp}$

(a) $\underset{\sim}{x} \in N(P)$, $\underset{\sim}{y} \in C(P) \Rightarrow \underset{\sim}{y} = P\underset{\sim}{w}$ for some $\underset{\sim}{w}$

$$\langle \underset{\sim}{x}, \underset{\sim}{y} \rangle = \langle \underset{\sim}{x}, P\underset{\sim}{w} \rangle \xrightarrow[\text{P Hermitian}]{=} \langle P\underset{\sim}{x}, \underset{\sim}{w} \rangle = \langle \underset{\sim}{0}, \underset{\sim}{w} \rangle = 0$$

So $\underset{\sim}{x} \in C(P)^{\perp}$

(b) Grab $\underset{\sim}{x} \in C(P)^{\perp}$. Look at

in $C(P)$ since $\underset{\sim}{x} \in C(P)^{\perp}$

$$\langle P\underset{\sim}{x}, P\underset{\sim}{x} \rangle \xrightarrow[\text{P Hermitian}]{=} \langle P^2\underset{\sim}{x}, \underset{\sim}{x} \rangle \xrightarrow[\text{P projector}]{=} \langle P\underset{\sim}{x}, \underset{\sim}{x} \rangle = 0$$

So $P\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \underset{\sim}{x} \in N(P)$

So $N(P) = C(P)^{\perp}$

($\Leftarrow$) Assume $N(P) = C(P)^\perp$

Grab $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{C}^n$ $\quad \underset{\sim}{u} = \underbrace{\underset{\sim}{u_1}}_{C(P)} + \underbrace{\underset{\sim}{u_2}}_{N(P)} \qquad \underset{\sim}{v} = \underbrace{\underset{\sim}{v_1}}_{C(P)} + \underbrace{\underset{\sim}{v_2}}_{N(P)}$

$$\langle P\underset{\sim}{u}, \underset{\sim}{v} \rangle = \langle P\underset{\sim}{u_1} + P\underset{\sim}{u_2}, \underset{\sim}{v_1} + \underset{\sim}{v_2} \rangle = \langle P\underset{\sim}{u_1}, \underset{\sim}{v_1} + \underset{\sim}{v_2} \rangle$$

$$= \langle \underset{\sim}{u_1}, \underset{\sim}{v_1} + \underset{\sim}{v_2} \rangle = \langle \underset{\sim}{u_1}, \underset{\sim}{v_1} \rangle + \underbrace{\langle \underset{\sim}{u_1}, \underset{\sim}{v_2} \rangle}_{\substack{C(P)\ N(P) \\ \text{orthogonal}}} = \langle \underset{\sim}{u_1}, \underset{\sim}{v_1} \rangle$$

$P\underset{\sim}{x} = \underset{\sim}{x}$

$\underset{\sim}{x} \in C(P)$

$$\langle \underset{\sim}{u}, P\underset{\sim}{v} \rangle = \langle \underset{\sim}{u_1} + \underset{\sim}{u_2}, P\underset{\sim}{v_1} + P\underset{\sim}{v_2} \rangle = \langle \underset{\sim}{u_1} + \underset{\sim}{u_2}, P\underset{\sim}{v_1} \rangle$$

$$= \langle \underset{\sim}{u_1} + \underset{\sim}{u_2}, \underset{\sim}{v_1} \rangle = \langle \underset{\sim}{u_1}, \underset{\sim}{v_1} \rangle + \underbrace{\langle \underset{\sim}{u_2}, \underset{\sim}{v_1} \rangle}_{\text{orthogonal}} = \langle \underset{\sim}{u_1}, \underset{\sim}{v_1} \rangle$$

So $\langle P\underset{\sim}{u}, \underset{\sim}{v} \rangle = \langle \underset{\sim}{u}, P\underset{\sim}{v} \rangle$ for <u>all</u> $\underset{\sim}{u}, \underset{\sim}{v}$ $\Rightarrow P$ Hermitian

Theorem 1.6.12 U subspace, A matrix w/ columns basis of U

$P = A(A^*A)^{-1}A^*$ is an orthogonal projector onto U

Proof $P^2 = [A(A^*A)^{-1}A^*][A(A^*A)^{-1}A^*]$

$= A(A^*A)^{-1}(A^*A)(A^*A)^{-1}A^* = A(A^*A)^{-1}A^* = P$

$\Rightarrow P$ projector

$N(P) = \{P\underset{\sim}{x} - \underset{\sim}{x} \mid \underset{\sim}{x} \in \mathbb{C}^n\}$ Lemma 1.6.3

Is $C(P)$ & $N(P)$ orthogonal?

$C(P) = C(A)$

$A^*(P\underset{\sim}{x} - \underset{\sim}{x}) = A^*P\underset{\sim}{x} - A^*\underset{\sim}{x} = A^*A(A^*A)^{-1}A^*\underset{\sim}{x} - A^*\underset{\sim}{x}$

$= A^*\underset{\sim}{x} - A^*\underset{\sim}{x}$

$= \underset{\sim}{0}$

rows columns of A span column space of P

element of N(P)

$C(P)$?

$P = A\underbrace{(A^*A)^{-1}A^*}_{\text{matrix}}$

$P\underset{\sim}{x} = A\underbrace{\underline{\qquad}\,\underset{\sim}{x}}_{\text{column vector}}$

= lin combination of columns of A

$C(P) \subseteq C(A)$

A has full rank

$\Rightarrow C(A) \subseteq C(P)$