Math 390 Thursday, April 8 Least-Squares

Drafts – April 18

Solve $A\underset{\sim}{x} = \underset{\sim}{b}$, but system has no solution.

$\iff \underset{\sim}{b} \notin C(A)$

Strategy: Find $\hat{\underset{\sim}{b}} \in C(A)$, $\hat{\underset{\sim}{b}}$ "close to" $\underset{\sim}{b}$.

For $\hat{\underset{\sim}{b}}$, set $A\hat{\underset{\sim}{x}} = \hat{\underset{\sim}{b}}$, ie $\hat{x}$ is solution to system, for $\hat{\underset{\sim}{b}}$ instead

Want $\hat{\underset{\sim}{b}} - \underset{\sim}{b}$ orthogonal to $C(A)$, $A\hat{\underset{\sim}{x}} - \underset{\sim}{b} \in (C(A))^{\perp} = N(A^*)$

$A^*(A\hat{\underset{\sim}{x}} - \underset{\sim}{b}) = \underset{\sim}{0}$

$A^*A\hat{\underset{\sim}{x}} - A^*\underset{\sim}{b} = \underset{\sim}{0}$

$A^*A\hat{\underset{\sim}{x}} = A^*\underset{\sim}{b}$ ← normal equations

$\hat{\underset{\sim}{x}} = \underbrace{(A^*A)^{-1}A^*}_{\text{pseudo-inverse}}\underset{\sim}{b}$

$\hat{r} = \hat{\underset{\sim}{b}} - \underset{\sim}{b}$ "residual"

(or $\underset{\sim}{b} - \hat{\underset{\sim}{b}}$)

("small" hopefully)

Example 4.1.2 Solve $A\underset{\sim}{x} = \underset{\sim}{b}$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} \quad \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \underset{\sim}{b} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$$

No solution (RREF on augmented)

$$\begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \notin \left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \right\rangle$$

Normal Equations

$$A^*A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ 5 & 10 \end{bmatrix}$$

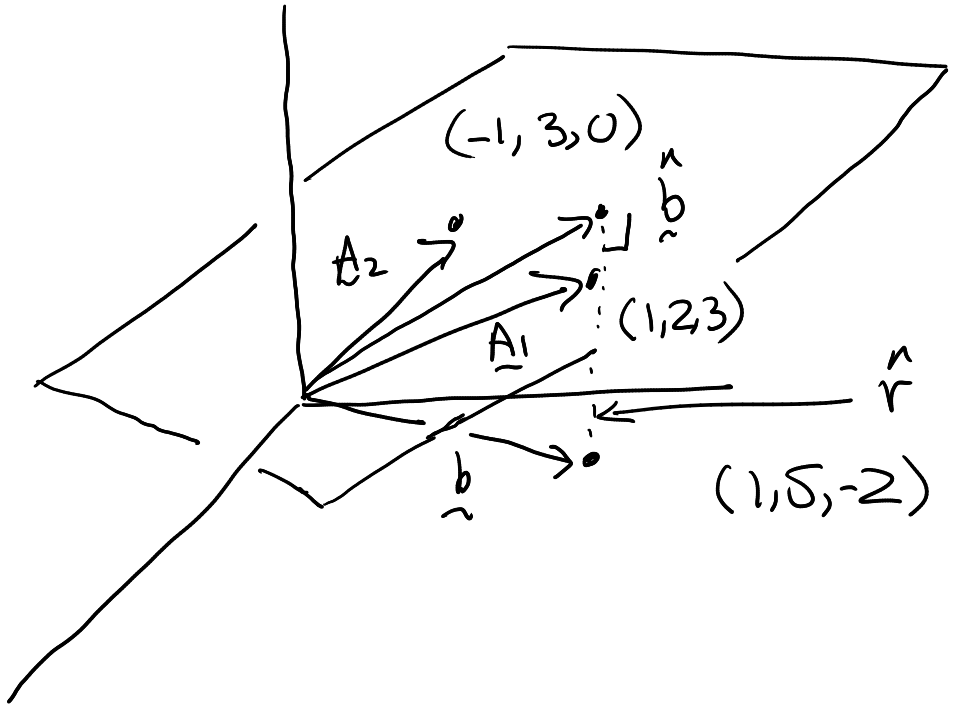
$$A^*\underset{\sim}{b} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$$

square, non singular

A has full rank

Solve $\begin{bmatrix} 14 & 5 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \end{bmatrix}$ $\longrightarrow$ $\hat{\underset{\sim}{x}} = \begin{bmatrix} -4/23 \\ 171/115 \end{bmatrix}$, $\hat{\underset{\sim}{b}} = A\hat{\underset{\sim}{x}} = \begin{bmatrix} -191/115 \\ 473/115 \\ -12/23 \end{bmatrix} \approx \begin{bmatrix} -1.5 \\ 4 \\ -1/2 \end{bmatrix}$

Geometry


(-1, 3, 0)
$\hat{b}$
$\underset{\sim}{A_2}$
$\underset{\sim}{A_1}$
(1,2,3)
$\hat{r}$
$\underset{\sim}{b}$
(1,5,-2)