Math 390 Friday, April 2 Least-Squares

Mon - Problems

Tue - Exam 2

Problem: Solve $A\underset{\sim}{x} = \underset{\sim}{b}$

Possibility: no solution.

Workaround: Replace $\underline{b}$ by $\hat{b}$, which **does** yield a solution.

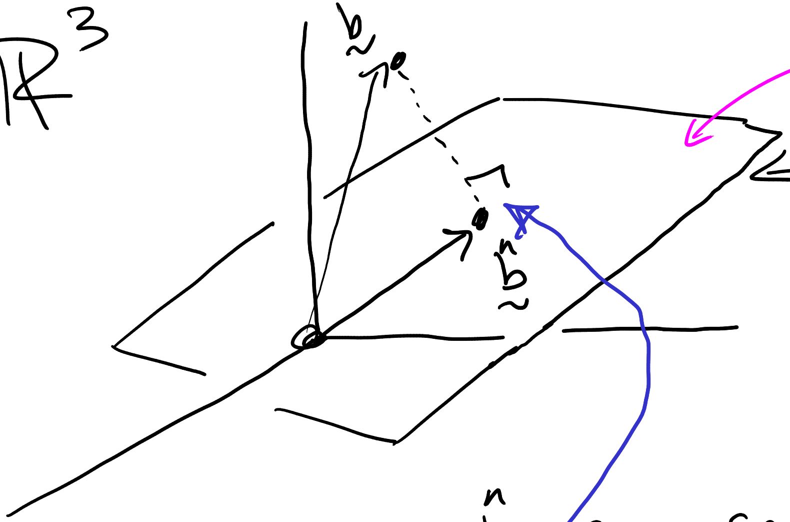
$A\underset{\sim}{x} = \underset{\sim}{b}$ no solution / inconsistent $\iff$ $\underset{\sim}{b} \notin C(A)$

Replace $\underset{\sim}{b}$ by $\hat{\underset{\sim}{b}}$, but want $\hat{\underset{\sim}{b}} \in C(A)$

Choose $\hat{b}$ closest to $\underset{\sim}{b}$.

Make $\| \underset{\sim}{b} - \hat{\underset{\sim}{b}} \|$ small

$\mathbb{R}^3$



plane: 2-D subspace

$C(A)$

$Ax = b$ ← $\mathbb{R}^3$

We want $\hat{b}$ so so that $b - \hat{b}$ orthogonal to $C(A)$

$$\hat{b} - b = Ax - b \in (C(A))^{\perp} = N(A^*)$$

$$A^*(Ax - b) = 0$$

$$A^*Ax - A^*b = 0$$

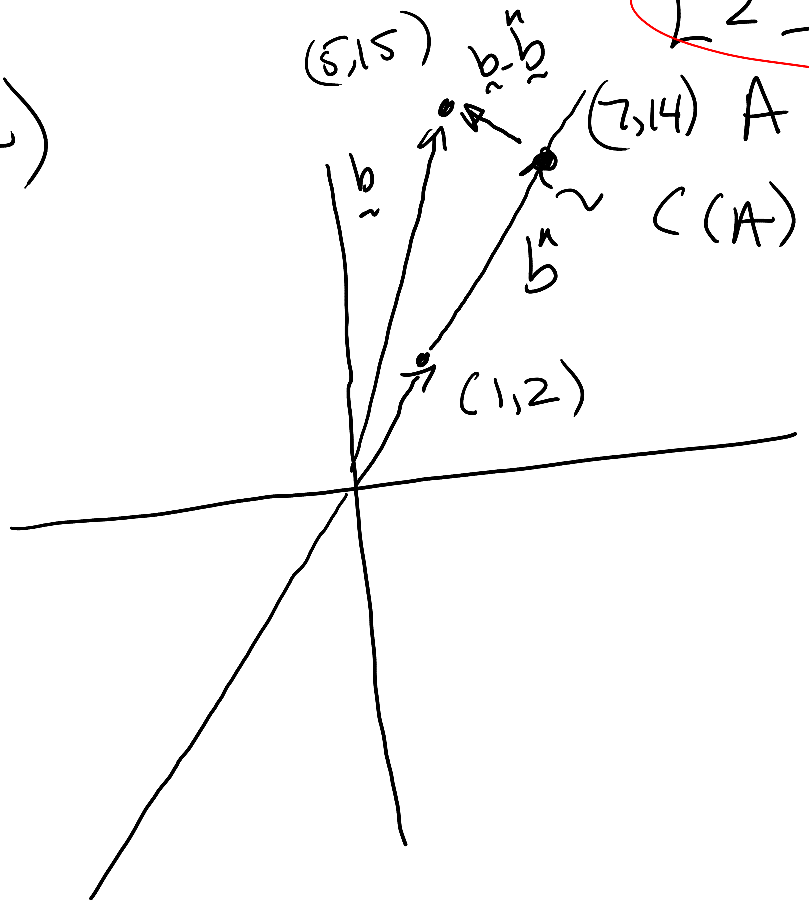
$$\underbrace{\boxed{A^*A}\, x = A^*b}_{\text{normal equations}}$$ ← solve this system for x

square, Hermitian, positive semidefinite.
If A has full rank, then A^*A is nonsingular

$$\Rightarrow x = \underbrace{(A^*A)^{-1}A^*}_{\text{pseudo inverse}}\, b$$

Example 4.1.1 Solve $\underset{A}{\begin{bmatrix}1\\2\end{bmatrix}}\underset{x}{\begin{bmatrix}x_1\end{bmatrix}} = \underset{b}{\begin{bmatrix}5\\15\end{bmatrix}}$ no solution

C(A)



Normal Equations

$$A^*A = \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5\end{bmatrix}_{1\times 1}$$

$$A^*b = \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}5\\15\end{bmatrix} = \begin{bmatrix}35\end{bmatrix}_{1\times 1}$$

$$A^*Ax = A^*b \Rightarrow \underset{1\times 1}{\begin{bmatrix}5\end{bmatrix}}\underset{1\times 1}{\begin{bmatrix}x_1\end{bmatrix}} = \underset{1\times 1}{\begin{bmatrix}35\end{bmatrix}}$$

$x_1 = 7$

"Check" our solution

$$Ax = \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}7\end{bmatrix} = \begin{bmatrix}7\\14\end{bmatrix} = \hat{b}$$

"Better" system

$$\left(\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}x_1\end{bmatrix} = \begin{bmatrix}7\\14\end{bmatrix}\right)$$

$$b - \hat{b} = \begin{bmatrix}5\\15\end{bmatrix} - \begin{bmatrix}7\\14\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$$

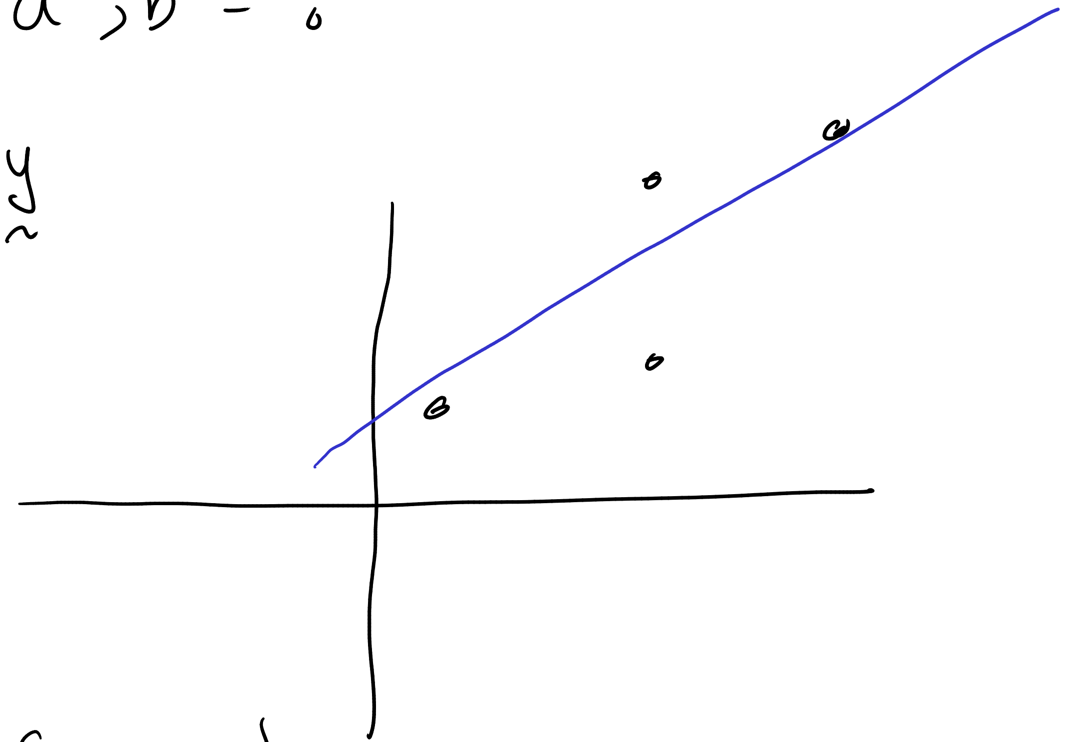
C(A) = line w/ slope 2

$\hat{b}-\hat{b}$ = vector w/ slope $-1/2$

Model : $y = a + bx$ $\quad a, b = ?$

observe x, measure $\underset{\sim}{y}$

x	y
37.3	21.2
9.8	7.3
12.6	9.1
14.9	6.4



There is no solution for a & b which provides $y = a + bx$ & matches the data