Math 390 Friday, March 26 OD/ Schur

Thu / Fri] Cholesky, Plus

Theorem OD A square

Unitary U, diagonal D so that

$U^* A U = D \iff A$ normal

Comments: • columns of U are orthonormal basis of eigenvector of A

• Diagonal elements of D are eigenvalues of A

Proof ($\Rightarrow$)

$U^*AU = D$

$\Rightarrow A = UDU^*$

$$
\begin{aligned}
A^*A &= (UDU^*)^*(UDU^*) \\
&= UD^*U^*UDU^* \\
&= U\boxed{D^*D}U^* \qquad \text{diagonal matrices commute} \\
&= U\boxed{DD^*}U^* \\
&= UDU^*UD^*U^*
\end{aligned}
$$

$$= UDU^* (UDU^*)^*$$

$$= AA^*$$

($\Leftarrow$) Know A is normal

Thm OBUTR $\Rightarrow$ Unitary U so that $U^*AU = T$ $\leftarrow$ upper-triangular

claim T is also normal

$$\begin{aligned}
T^*T &= (U^*AU)^*(U^*AU) \\
&= U^*A^*UU^*AU \\
&= U^*\boxed{A^*A}U \\
&= U^*\boxed{AA^*}U \\
&= U^*AUU^*A^*U \\
&= (U^*AU)(U^*AU)^* \\
&= TT^*
\end{aligned}$$

Fact $TT^* - T^*T = 0$

$$0 = [TT^* - T^*T]_{ii} = [TT^*]_{ii} - [T^*T]_{ii}$$

$$= \sum_{k=1}^{n} [T]_{ik}[T^*]_{ki} - \sum_{k=1}^{n} [T^*]_{ik}[T]_{ki}$$

$$= \sum_{k=1}^{n} [T]_{ik}\overline{[T]_{ik}} - \sum_{k=1}^{n} \overline{[T]_{ki}}\,[T]_{ki}$$

$$= \sum_{k=i}^{n} [T]_{ik}\overline{[T]_{ik}} - \sum_{k=1}^{i} \overline{[T]_{ki}}\,[T]_{ki}$$

$$= \sum_{k=i}^{n} \left|[T]_{ik}\right| - \sum_{k=1}^{i} \left|[T]_{ki}\right|$$

T upper-triangular

positive

$(a+bi)(a-bi) = a^2+b^2$

$|a+bi| = \sqrt{a^2+b^2}$

Now determine that entries of T, above the diagonal, are all zero, starting with row $i=1$

$$0 = \sum_{k=1}^{n} |[T]_{1k}| - \sum_{k=1}^{1} |[T]_{k1}| = \sum_{k=1}^{n} |[T]_{1k}| - |[T]_{11}|$$

$$= \sum_{k=2}^{n} |[T]_{1k}|$$ ← sum of positive quantities

$$\Rightarrow [T]_{12} = 0, \; [T]_{13} = 0, \; \ldots, \; [T]_{1n} = 0$$

First row is all zeros (except diagonal)

Again w/ $i=2$

$$0 = \sum_{k=2}^{n} |[T]_{2k}| - \sum_{k=1}^{2} |[T]_{k2}|$$

$$= \sum_{k=2}^{n} |[T]_{2k}| - \cancel{|[T]_{12}|} - |[T]_{22}|$$

$\cancel{|[T]_{12}|} \to 0$

$$= \sum_{k=3}^{n} |[T]_{2k}|$$ ← sum of positive quantities

$$\Rightarrow [T]_{23} = 0, \; [T]_{24} = 0, \; \ldots, \; [T]_{2n} = 0$$

Second row is zeros to the right of the diagonal

Repeat, **in order**, for each subsequent row