Math 390 Monday, March 15 P.S.D. + SVD

Exam 2 : April 6

Theorem 1.8.3 A Hermitian

A is p.s.d $\iff$ every eigenvalue of A has $\lambda \geq 0$.

Proof

($\Rightarrow$) $\underset{\sim}{x}$ eigenvector of A for λ.

$$\lambda \underbrace{\langle \underset{\sim}{x}, \underset{\sim}{x} \rangle}_{>0 \text{ since } \underset{\sim}{x} \neq \underset{\sim}{0} \text{ (Theorem PIP)}} = \langle \underset{\sim}{x}, \lambda \underset{\sim}{x} \rangle = \langle \underset{\sim}{x}, A\underset{\sim}{x} \rangle \underset{\text{p.s.d.}}{\geq} 0 \quad \Big| \quad \Rightarrow \lambda \geq 0$$

($\Leftarrow$) A Hermitian $\Rightarrow$ A normal

$\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n\}$ orthonormal basis of $\mathbb{C}^n$, eigenvectors of A.

$$\langle \underset{\sim}{x}, A\underset{\sim}{x} \rangle = \left\langle \sum_i a_i \underset{\sim}{x_i}, A \sum_i a_i \underset{\sim}{x_i} \right\rangle$$

$$\langle \underset{\sim}{x}, \underset{\sim}{y}+\underset{\sim}{z} \rangle = \langle \underset{\sim}{x}, \underset{\sim}{y} \rangle + \langle \underset{\sim}{x}, \underset{\sim}{z} \rangle$$

$$= \left\langle \sum_i a_i \underset{\sim}{x_i}, \sum a_i A \underset{\sim}{x_i} \right\rangle$$

$$= \left\langle \sum_i a_i \underset{\sim}{x_i}, \sum_i a_i \lambda_i \underset{\sim}{x_i} \right\rangle$$

$$= \sum_i \sum_j \langle a_i \underset{\sim}{x_i}, a_j \lambda_j x_j \rangle$$

$$= \sum_i \langle a_i \underset{\sim}{x_i}, a_i \lambda_i \underset{\sim}{x_i} \rangle + \sum_{i \neq j} \langle a_i x_i, a_j \lambda_j \underset{\sim}{x_j} \rangle$$

$$= \sum_i \bar{a}_i a_i \lambda_i \underbrace{\langle \underset{\sim}{x_i}, \underset{\sim}{x_i} \rangle}_{\to 1} + \sum_{i \neq j} \bar{a}_i a_j \lambda_j \underbrace{\langle \underset{\sim}{x_i}, \underset{\sim}{x_j} \rangle}_{\to 0}$$

$$= \sum_i \underbrace{\bar{a}_i a_i}_{\text{positive}} \underbrace{\lambda_i}_{\text{positive}} \geq 0 \qquad (\text{for } \underline{\underline{\text{all}}}\ \underset{\sim}{x})$$

$$(a+bi)(a-bi) = a^2 + b^2$$

SVD

Theorem 2.3.1 A $m \times n$ matrix, A^*A rank r

$\lambda_1, \lambda_2, \dots \lambda_p$ non zero eigenvalues of A^*A

$\rho_1, \rho_2, \dots, \rho_q$ non zero eigenvalues of AA^* **Then**

1) $p = q$

2) $\lambda_1 = \rho_1, \lambda_2 = \rho_2, \dots, \lambda_p = \rho_p$ (suitably ordered)

3) $\alpha_{A^*A}(\lambda_i) = \alpha_{AA^*}(\rho_i)$ (suitably ordered)

4) rank of A^*A = rank of AA^*

5) orthonormal basis of $\mathbb{C}^n$ $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \dots, \underset{\sim}{x}_n\}$, eigenvectors of A^*A
orthonormal basis of $\mathbb{C}^m$ $\{\underset{\sim}{y}_1, \underset{\sim}{y}_2, \dots, \underset{\sim}{y}_m\}$, eigenvectors of AA^*
Let $\underset{\sim}{x}_{r+1}, \dots \underset{\sim}{x}_n$ be the eigenvectors of A^*A for zero eigenvalue
Let δ_i, $1 \le i \le r$ be the <u>nonzero</u> eigenvalues of A^*A.

Then $A\underset{\sim}{x}_i = \sqrt{\delta_i}\,\underset{\sim}{y}_i$, $1 \le i \le r$, $A\underset{\sim}{x}_i = \underset{\sim}{0}$, $r+1 \le i \le n$, $\underset{\sim}{y}_i$ eigenvector for zero of AA^*, $r+1 \le i \le m$

<u>Proof</u>

$\underset{\sim}{x}$ eigenvector of A^*A for λ

Then $\underline{A\underset{\sim}{x}}$ is an eigen vector of $\underline{AA^*}$ for λ.

$$\boxed{(AA^*)(A\underset{\sim}{x}) = A(A^*A\underset{\sim}{x}) = A(\lambda\underset{\sim}{x}) = \lambda(A\underset{\sim}{x})}$$

Eigenvalues of A^*A are the eigenvalues of AA^*.