Math 390 Friday, March 12 Normal, Positive Semi-Definite, SVD

Proof from last time, need to assume $\underset{\sim}{x} \in \mathbb{R}^n$ (not $\mathbb{C}^n$)

Theorem Given $\underset{\sim}{x}$, define $\underset{\sim}{v} = \underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{e_1}$, P Householder for $\underset{\sim}{v}$,

then $P\underset{\sim}{x} = \mp \|\underset{\sim}{x}\| \underset{\sim}{e_1}$ (P "zeros out" $\underset{\sim}{x}$)

Proof

$$\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = (\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{e_1})^* (\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{e_1})$$

$$= (\underset{\sim}{x}^* \pm \|\underset{\sim}{x}\| \underset{\sim}{e_1}^*)(\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{e_1})$$

$$= \underset{\sim}{x}^*\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{x}^* \underset{\sim}{e_1} \pm \|\underset{\sim}{x}\| \underset{\sim}{e_1}^* \underset{\sim}{x} + \|\underset{\sim}{x}\|^2 \underset{\sim}{e_1}^* \underset{\sim}{e_1}$$

$$= \underset{\sim}{x}^*\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{x}^* \underset{\sim}{e_1} \pm \|\underset{\sim}{x}\| \underset{\sim}{e_1}^* \underset{\sim}{x} + \underset{\sim}{x}^*\underset{\sim}{x}\,(1)$$

$$= 2\underset{\sim}{x}^*\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \overline{\underset{\sim}{e_1}^* \underset{\sim}{x}} \pm \|\underset{\sim}{x}\| \underset{\sim}{e_1}^* \underset{\sim}{x}$$

$$= 2\underset{\sim}{x}^*\underset{\sim}{x} \pm 2\|\underset{\sim}{x}\| \underset{\sim}{e_1}^* \underset{\sim}{x}$$

$$= 2(\underset{\sim}{x}^* \pm \|\underset{\sim}{x}\| \underset{\sim}{e_1}^*)\underset{\sim}{x} = 2(\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{e_1})^* \underset{\sim}{x} = \underline{2\underset{\sim}{v}^* \underset{\sim}{x}}$$

1.7 Normal Matrices

Defn A square, we say A is <u>normal</u> if $A^*A = AA^*$.

Examples diagonal matrix, Hermitian (self-adjoint) (symmetric), unitary ($Q^*Q = I$, so $QQ^* = I$), skew-symmetric ($A^t = -A$)

Example 1.7.2 $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ normal, not one of the above

Theorem OD (FCLA)

$$\underbrace{A = U^* D U}_{\text{orthonormal diagonalization}} \iff A \text{ normal}$$

D: diagonal

U: unitary

1.8 Positive Semi-Definite Matrices

<u>Defn</u> A p.s.d. if A Hermitian, and for all $\underset{\sim}{x}$ $\langle \underset{\sim}{x}, A\underset{\sim}{x} \rangle \geq 0$

positive definite $>$

negative semi-definite $\leq$

negative definite $<$

<u>Therem</u> Suppose A is $m \times n$ matrix then $\underset{n\times n}{A^*A}$ & $\underset{m\times m}{AA^*}$ are p.s.d.

<u>Proof</u> Check AA^* Hermitian,

$$(AA^*)^* = (A^*)^* A^* = AA^*.$$

Check $\langle \underset{\sim}{x}, (AA^*)\underset{\sim}{x} \rangle = \langle \underset{\sim}{x}, A(A^*\underset{\sim}{x}) \rangle = \langle A^*\underset{\sim}{x}, A^*\underset{\sim}{x} \rangle \geq 0$ for all $\underset{\sim}{x} \in \mathbb{C}^m$

inner-product property

Theorem PIP

$\langle \underset{\sim}{u}, \underset{\sim}{u} \rangle \in \mathbb{R}$ $\quad \alpha \in \mathbb{C}$, $\alpha = a+bi$

u^*u $\quad \bar{\alpha}\alpha = a^2+b^2$

$w = f(x,y,z)$ $\qquad \vec{x} = (x,y,z)^t$

$= f(\vec{x})$

$$f(\vec{x}) = f(\vec{x}_0) + \cancel{\vec{\nabla} f(x_0)\cdot\vec{x}} + \vec{x}^t H \vec{x}$$

0

$\langle \vec{x}, H\vec{x} \rangle$

max/min, critical point

$\vec{\nabla} f(x_0) = \vec{0}$

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

x_0 is a min, want to be always positive

$D = f_{xx} f_{yy} - f_{xy}^2$

$f_{xx} > 0$