Math 390 Thursday, March 11 Householder Reflector, QR Decomposition

Given $\underset{\sim}{v} \Rightarrow P = I_n - \frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v}\underset{\sim}{v}^*$

Facts P Hermitian $(P = P^*)$, P unitary $(P^* P = I_n)$

$(\Rightarrow P^2 = I)$

Theorem Given $\underset{\sim}{x}$, define $\underset{\sim}{v} = \underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{e}_1$, P Householder for $\underset{\sim}{v}$,

$\left[\begin{smallmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{smallmatrix}\right]$

then $P\underset{\sim}{x} = \mp \|\underset{\sim}{x}\| \underset{\sim}{e}_1$ (P "zeros out" $\underset{\sim}{x}$)

Proof

$$\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle = (\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{e}_1)^* (\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{e}_1)$$

$$= (\underset{\sim}{x}^* \pm \|\underset{\sim}{x}\| \underset{\sim}{e}_1^*)(\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{e}_1)$$

$$= \underset{\sim}{x}^*\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{x}^* \underset{\sim}{e}_1 \pm \|\underset{\sim}{x}\| \underset{\sim}{e}_1^* \underset{\sim}{x} + \|\underset{\sim}{x}\|^2 \underset{\sim}{e}_1^* \underset{\sim}{e}_1$$

$$= x^* \underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{x}^* \underset{\sim}{e}_1 \pm \|\underset{\sim}{x}\| \underset{\sim}{e}_1^* \underset{\sim}{x} + x^* x (1)$$

$$= 2x^* \underset{\sim}{x} \qquad\qquad = 2\underset{\sim}{v}^* \underset{\sim}{x}$$

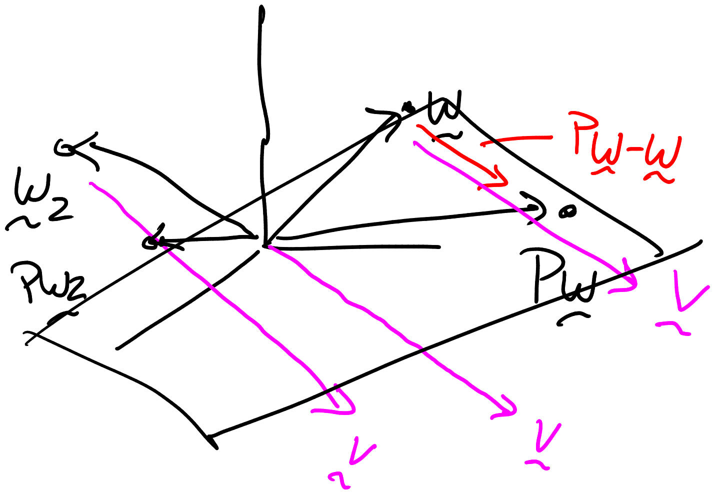
$$P\underset{\sim}{x} = \left(I_n - \frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v} \underset{\sim}{v}^* \right) \underset{\sim}{x}$$

$$= \underset{\sim}{x} - \frac{2}{2v^* \underset{\sim}{x}} v v^* \underset{\sim}{x}$$

$$= \underset{\sim}{x} - \underset{\sim}{v} = \underset{\sim}{x} - (\underset{\sim}{x} \pm \|\underset{\sim}{x}\| \underset{\sim}{e}_1) = \mp \|\underset{\sim}{x}\| \underset{\sim}{e}_1$$

Why "reflector"? Comparing $\underset{\sim}{w}$ & $P\underset{\sim}{w}$

$$P\underset{\sim}{w} - \underset{\sim}{w} = I_n \underset{\sim}{w} - \frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} v v^* \underset{\sim}{w} - \underset{\sim}{w}$$



$$= -\frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v} \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = -\frac{2\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v}$$

Scalar multiple of $\underset{\sim}{v}$

What points $\underset{\sim}{w}$ do not move at all?

When is $\dfrac{-2\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} = 0$? Points w/ $\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = 0$

$\underset{\sim}{w} \perp \underset{\sim}{v}$.