Math 390 Monday, March 8 QR & Householder

$$A = [\underset{\sim}{A_1} \mid \underset{\sim}{A_2} \mid \cdots \mid \underset{\sim}{A_n}] \quad m \times n$$

$$\underset{m\times n}{[\underset{\sim}{A_1} \mid \underset{\sim}{A_2} \mid \cdots \mid \underset{\sim}{A_n}]} \underset{n\times n}{\begin{bmatrix} x & x & x & \cdots & x \\ 0 & x & x & \cdots & x \\ 0 & 0 & x & \cdots & \vdots \\ \vdots & \vdots & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x \end{bmatrix}} = \underset{m\times n}{[\underbrace{\underset{\sim}{Q_1} \mid \underset{\sim}{Q_2} \mid \underset{\sim}{Q_3} \mid \cdots \mid \underset{\sim}{Q_n}}_{\text{orthonormal set}}]}$$

originals

upper triangular

orthonormal set

via Gram-Schmidt

$$A\hat{R} = Q$$

If A has full rank, then $\hat{R}$ has non zero diagonal $\Rightarrow$ $\hat{R}$ invertible ($\hat{R} = (R)^{-1}$)

$$A = QR$$

A square: of size m

$$\underset{m\times m}{A} = \underset{m\times m}{Q}\ \underset{m\times n}{R}$$

Q unitary! $Q^* = \bar{Q}^t$

unitary: $Q^*Q = I \Leftrightarrow Q^{-1} = Q^*$ ($\mathbb{R}$ orthogonal)

Solving systems w/ QR

Solve $A\underset{\sim}{x} = \underset{\sim}{b}$ A square, full rank

$QR\underset{\sim}{x} = \underset{\sim}{b}$

$R\underset{\sim}{x} = Q^{-1}\underset{\sim}{b}$

$R\underset{\sim}{x} = Q^{*}\underset{\sim}{b}$ EZ

columns of Q^* are norm 1, behave well numerically, no mismatch in magnitudes

upper triangular $\Rightarrow$ back solve

1.5 Reflectors   Householder Matrix

Defn Given $\underset{\sim}{v} \in \mathbb{C}^n$ for Householder matrix (of $\underset{\sim}{v}$) is $P = I_n - \boxed{\frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle}}\,\boxed{\underset{\sim}{v}\,\underset{\sim}{v}^*}$

scalar $n \times n$ matrix

the Householder vector $\Rightarrow P$ $n \times n$ matrix

$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle$ inner product, scalar

$\langle \underset{\sim}{v}, \underset{\sim}{w} \rangle = \underset{1\times n}{\underset{\sim}{v}^*}\ \underset{n\times 1}{\underset{\sim}{w}}$ 1×1 matrix, scalar

$\underset{n\times 1}{\underset{\sim}{v}}\ \underset{1\times n}{\underset{\sim}{w}^*}$ "outer product"

$\underline{\underline{n \times n}}$ matrix   rank 1 matrix, all columns multiple of $\underset{\sim}{v}$

Ex $\underset{\sim}{v} = [1\ 2\ 3]^t$, $\underset{\sim}{w} = [10\ 20\ 2]^t$ $\underset{\sim}{v}\underset{\sim}{w}^* = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [10\ 20\ 2] = \begin{bmatrix} 10 & 20 & 2 \\ 20 & 40 & 4 \\ 30 & 60 & 6 \end{bmatrix}$

<u>Lemma</u> P is Hermitian (self-adjoint) $P = P^*$ ($\mathbb{R}$: symmetric)

<u>Proof</u>
$$P^* = \left(I_n - \frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v}\underset{\sim}{v}^*\right)^*$$
$$= I_n^* - \frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} (\underset{\sim}{v}\underset{\sim}{v}^*)^*$$
$$= I_n - \frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v}\underset{\sim}{v}^* = P \ !$$

<u>Lemma</u> P is unitary

$$P^*P = PP = \left(I_n - \frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v}\underset{\sim}{v}^*\right)\left(I_n - \frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v}\underset{\sim}{v}^*\right)$$ FOIL

$$= I_n I_n - \frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v}\underset{\sim}{v}^* - \frac{2}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v}\underset{\sim}{v}^* + \frac{4}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle^2} \underset{\sim}{v}\underline{\underset{\sim}{v}^*\underset{\sim}{v}}\underset{\sim}{v}^*$$

$$= I_n - \frac{4}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} \underset{\sim}{v}\underset{\sim}{v}^* + \frac{4}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle^2} \underline{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle}\, vv^* = I_n - \frac{4}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} vv^* + \frac{4}{\langle \underset{\sim}{v}, \underset{\sim}{v} \rangle} vv^* = I_n$$