Math 390 Monday, March 1 Minimal polynomial, Rational Canonical Form

Tue - Problem Session (Problem 17)

Defn Given A $n \times n$ matrix, $m(x)$ monic polynomial of least degree w/ $m(A) = \mathcal{O}$ then $m(x)$ is minimal polynomial of A.

Theorem Exists, degree $\leq n$.

Proof Cayley-Hamilton Theorem, $P_A(A) = \mathcal{O}$.

Computing minimal polynomial

1. $\{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots, \underset{\sim}{v_n}\}$ basis of $\mathbb{C}^n$
2. Least degree polynomial w/ $P_i(A)\,\underset{\sim}{v_i} = \underset{\sim}{0}$
3. Minimal polynomial = least common multiple $p_1, p_2, \dots, p_n$.

Ex $C(2-3x+x^2) = \underset{2\times 2}{\begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}}$ companion matrix

monic

Rational Canonical Form

matrix, block diagonal, each block a companion matrix of $g_1, g_2, \ldots, g_t$, with $g_1 | g_2 | g_3 | \ldots | g_t$

Theorem Every linear transformation has a basis where the matrix representation is in rational canonical form.

Ex
$$\begin{bmatrix} \begin{bmatrix} & & 2 \\ 1 & & -4 \\ & 1 & 6 \end{bmatrix} & \\ & \begin{bmatrix} & & & 3 \\ 1 & & & -2 \\ & 1 & & 5 \\ & & 1 & 2 \end{bmatrix} \end{bmatrix}$$
in rational canonical form

Fact g_t is the minimal polynomial

Computing Rational Canonical Form

- Smith normal form
- structure theorem for finitely-generated modules over a principal ideal domain (PID)
- elementary divisors
- invariant factors

] often confused

no lesser degree

(1) Find minimal polynomial, and a maximal vector $\underline{v}$ $m(A)\underline{v} = \underline{0}$

Cyclic subspace $V = \langle \{ \underline{v}, A\underline{v}, A^2\underline{v}, \dots A^{m-1}\underline{v} \} \rangle$ ← A-invariant

(2) $T: V \to V$, W T-invariant

Define $T^* : V/W \to V/W$ in the natural way.

T^* is well-defined.

(3) Next companion matrix is minimal polynomial of T^*