Math 390 Monday January 25 Representations

FCLA Chapter R

Tue - Problems

Thu - Section EE (FCLA)

<u>Vector Representation</u>

V - vector space, $B = \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \ldots, \underset{\sim}{v_n}\}$ basis of V

Grab $\underset{\sim}{v} \in V$, $\underset{\sim}{v} = a_1\underset{\sim}{v_1} + a_2\underset{\sim}{v_2} + \cdots + a_n\underset{\sim}{v_n}$ ← Theorem VRRB

then $\rho_B(\underset{\sim}{v}) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$

<u>Fact</u> $\rho_B : V \to \mathbb{F}^n$ is an invertible linear transformation

isomorphism

ρ_B = "coordinatization" ρ_B^{-1} = linear combination (of B)

Ex $V = P_2$ $\qquad \underset{\sim}{v} = 3+4x-2x^2$

$B = \{1, x, x^2\}$ $\qquad \underset{\sim}{v} = 3(1) + 4(x) + (-2)(x^2)$

$$\rho_B(\underset{\sim}{v}) = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} \in \mathbb{C}^3$$

$C = \{1, 1+x, 1+x+x^2\}$ $\qquad \underset{\sim}{v} = -1(1) + 6(1+x) + -2(1+x+x^2)$

$$\rho_C(\underset{\sim}{v}) = \begin{bmatrix} -1 \\ 6 \\ -2 \end{bmatrix} \in \mathbb{C}^3$$

Fact $\qquad P_2 \cong \mathbb{C}^3$

Matrix Representations of Linear Transformations

Defn $T : U \to V$ w/ bases B & C, $B = \{\underset{\sim}{u}_1, \underset{\sim}{u}_2, \dots, \underset{\sim}{u}_n\}$

$$M^T_{B,C} = \left[\rho_C(T(\underset{\sim}{u}_1)) \,\middle|\, \rho_C(T(\underset{\sim}{u}_2)) \,\middle|\, \cdots\cdots \,\middle|\, \rho_C(T(\underset{\sim}{u}_n)) \right]$$

Ex $T : P_2 \to M_{22}$ $\quad T(a+bx+cx^2) = \begin{bmatrix} 2a-b+c & a+3b+2c \\ 5a+6c & 9a-2b+c \end{bmatrix}$

$B = \{1, x, x^2\}$, $C = \left\{ \begin{bmatrix} 1&0\\0&0 \end{bmatrix}, \begin{bmatrix} 0&1\\0&0 \end{bmatrix}, \begin{bmatrix} 0&0\\1&0 \end{bmatrix}, \begin{bmatrix} 0&0\\0&1 \end{bmatrix} \right\}$

$$\rho_C(T(1)) = \rho_C\left(\begin{bmatrix} 2&1\\5&9 \end{bmatrix} \right) = \rho_C\left(2\begin{bmatrix} 1&0\\0&0 \end{bmatrix} + 1\begin{bmatrix} 0&1\\0&0 \end{bmatrix} + 5\begin{bmatrix} 0&0\\1&0 \end{bmatrix} + 9\begin{bmatrix} 0&0\\0&1 \end{bmatrix} \right) = \begin{bmatrix} 2\\1\\5\\9 \end{bmatrix}$$

$$\rho_C(T(x)) = \rho_C\left(\begin{bmatrix} -1&3\\0&-2 \end{bmatrix} \right) = \begin{bmatrix} -1\\3\\0\\-2 \end{bmatrix}$$

$$\rho_C(T(x^2)) = \rho_C\left(\begin{bmatrix} 1&2\\6&1 \end{bmatrix} \right) = \begin{bmatrix} 1\\2\\6\\1 \end{bmatrix}$$

$$M^T_{B,C} = \begin{bmatrix} 2&-1&1\\1&3&2\\5&0&6\\9&-2&1 \end{bmatrix}$$

Theorem FTMR

matrix-vector product

$$\underbrace{\rho_C(T(\underset{\sim}{u}))}_{\text{col vector}} = \underbrace{M^T_{B,C}}_{\text{matrix}} \; \underbrace{\rho_B(\underset{\sim}{u})}_{\text{col. vector}}$$

OR

$$T(\underset{\sim}{u}) = \rho_C^{-1}\left(M^T_{B,C}\,\rho_B(\underset{\sim}{u})\right)$$

$$\begin{tikzcd}
\underset{\sim}{u} \arrow[r, "\rho_B"] \arrow[d, "T"'] & \rho_B(\underset{\sim}{u}) \arrow[d, "M^T_{B,C}"] \\
T(\underset{\sim}{u}) \arrow[r, shift right, "\rho_C"'] & M^T_{B,C}\,\rho_B(\underset{\sim}{u}) = \rho_C(T(u)) \arrow[l, shift right, "\rho_C^{-1}"']
\end{tikzcd}$$

equal

$$
\begin{tikzcd}
& U \arrow[r, "T"] \arrow[dl, dash, dashed] \arrow[dd, "\rho_B"] & V \arrow[dr, dash, dashed] \arrow[dd, "\rho_C"] & \\
\mathcal{K}(T) \arrow[dd, "\rho_B|_{\mathcal{K}(T)}"] & & & \mathcal{R}(T) \arrow[dd, "\rho_C|_{\mathcal{R}(T)}"] \\
& \mathbb{C}^n \arrow[r, "M^T_{B,C}"] \arrow[dl, dash, dashed] & \mathbb{C}^m \arrow[dr, dash, dashed] & \\
\mathcal{N}(M^T_{B,C}) & & & \mathcal{C}(M^T_{B,C})
\end{tikzcd}
$$

Figure KRI