Class—FCLA IS

Advanced Linear Algebra

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An 8×8 matrix, carefully engineered,

$$A = \begin{bmatrix} -14 & -2 & 3 & -15 & 3 & -6 & 4 & 3 \\ -6 & -7 & -10 & 21 & 7 & 8 & 3 & -9 \\ -24 & -17 & -11 & 23 & 17 & 14 & 6 & -13 \\ 19 & -3 & -10 & 37 & 2 & 17 & -5 & -10 \\ -21 & -15 & -12 & 20 & 17 & 11 & 6 & -11 \\ -30 & -5 & 4 & -24 & 5 & -9 & 8 & 4 \\ -37 & -11 & -1 & -14 & 11 & -5 & 12 & -1 \\ 22 & -1 & -8 & 32 & 0 & 16 & -6 & -6 \end{bmatrix}.$$

```
A = matrix(QQ, [
[-14, -2, 3, -15, 3, -6, 4, 3],
[-6, -7, -10, 21, 7, 8, 3, -9],
[-24, -17, -11, 23, 17, 14, 6, -13],
[19, -3, -10, 37, 2, 17, -5, -10],
[-21, -15, -12, 20, 17, 11, 6, -11],
[-30, -5, 4, -24, 5, -9, 8, 4],
[-37, -11, -1, -14, 11, -5, 12, -1],
[22, -1, -8, 32, 0, 16, -6, -6]
])
A
```

We saw before that $\lambda = 2$ and $\lambda = 3$ were potential eigenvalues, as roots of a polynomial $p(x) = (x-2)^3(x-3)^3$. We confirmed that they are indeed both eigenvalues by checking that $A - \lambda I_8$ is a singular matrix.

Demonstration 1 We will now investigate *generalized* eigenspaces for each. We can verify much of Theorem NSPM by looking at the dimensions of null spaces (nullities) of powers of $A - \lambda I_8$.

```
[((A-2)^i).nullity() for i in range(9)]
```

```
[((A-3)^i).nullity() for i in range(9)]
```

Notice that each sequence "tops out" at the third power. It is a coincidence that it is the same power for each eigenvalue (because this is a small example). It is actually not a great surprise (but not a theorem) that the polynomial we found has each factor to the third power.

The algebraic multiplicities are 5 and 3, which sum to $n = 8$. This will be a theorem that we have to work very hard to get.

Demonstration 2 Build each generalized eigenspace using `.right_kernel(basis='pivot')`, and do it twice. Once with the smallest possible power and once with the power $n = 8$. Notice that Sage will let us test the equality of these subspaces.

Demonstration 3 Get a basis for one generalized eigenspace, build a vector in the subspace with a very arbitrary linear combination, multiply by A, and test for membership in the subspace. This is demonstrating that the generalized eigenspace is an invariant subspace of A.