1. Consider the function \( T: \mathbb{C}^2 \to \mathbb{C}^2 \) below. (35 points)

\[
T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a - b \\ a + 3b \end{bmatrix}
\]

(a) Prove that \( T \) is a linear transformation.

(b) Is \( T \) injective? Why or why not?
Consider the linear transformation $T: S_{22} \to P_2$ defined below, where $S_{22}$ is the vector space of $2 \times 2$ symmetric matrices, and $P_2$ is the vector space of polynomials of degree at most 2. (35 points)

$T \left( \begin{bmatrix} a & b \\ b & c \end{bmatrix} \right) = (3a + b + 5c) + (2a + b + 4c)x + (-3a + 4b + 5c)x^2$

(a) Compute the kernel of $T$, $\mathcal{K}(T)$.

(b) Compute the range of $T$, $\mathcal{R}(T)$.

(c) The rank and nullity of $T$ obey a basic relationship. Say what this relationship is, and verify it for $T$.

(d) Compute the preimage of $2 + x - 7x^2$, $T^{-1}(2 + x - 7x^2)$. 


3. For the linear transformation $R: M_{22} \to \mathbb{C}^3$ find a specific element of the codomain with an empty pre-image, demonstrating that $R$ is not surjective. ($M_{22}$ is the vector space of $2 \times 2$ matrices.) (15 points)

$$R\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a + 2c + 2d \\ a - b + 3c + d \\ b - c + d \end{bmatrix}$$

4. Suppose that $S: U \to V$ and $T: V \to W$ are linear transformations and each is injective. Prove that their composition, $T \circ S$ is invertible. (15 points)