

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers, unless explicitly suggested in the problem's statement. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Consider the matrix  $A$  below. (35 points)

- (a) Compute some potential eigenvalues of  $A$ . Eventually, you will want to do this in a way that you find all of the eigenvalues of  $A$ , so be thorough. You may use Sage to factor polynomials.

$$A = \begin{bmatrix} 5 & 1 & -5 & 2 \\ 3 & 3 & -4 & 3 \\ 3 & 0 & -2 & 2 \\ -3 & -1 & 4 & -1 \end{bmatrix}$$

- (b) For each of your potential eigenvalues compute the traditional eigenspace. Explain how this allows you to now be certain which of your results in part (a) are definitely eigenvalues of  $A$ .

- (c) For each eigenvalue, compute a generalized eigenspace and an algebraic multiplicity. Explain how you can now be certain that you have computed every eigenvalue of  $A$ .

- (d) What is the characteristic polynomial of  $A$ ?



2. Consider the matrix  $B$  below. (25 points)

- (a) Use Sage's `.eigenvalues()` method to compute the eigenvalues of  $B$ . Then, by simply row-reducing the right matrices, determine the geometric and algebraic multiplicity of each eigenvalue.

$$B = \begin{bmatrix} 8 & 20 & -10 & 20 \\ -5 & -2 & -5 & -20 \\ -5 & 0 & -7 & -20 \\ 0 & -5 & 5 & 3 \end{bmatrix}$$

- (b) Explain how you now know that  $B$  is diagonalizable.

- (c) Again, just by row-reducing the right matrices, find a nonsingular matrix  $S$  which will diagonalize  $B$ . Give the resulting diagonal matrix  $D$ .

3. Suppose that 0 is an eigenvalue of  $A$ . Prove that  $A$  is singular. (10 points)



4. Prove the transitive property of an equivalence relation for similarity. That is, suppose  $A$  and  $B$  are similar matrices, and  $B$  and  $C$  are similar matrices. Conclude that  $A$  and  $C$  are similar matrices. (15 points)

5. Suppose that  $A$  is an  $m \times n$  matrix. Prove that the column space of  $A$ ,  $\mathcal{C}(A)$ , is an  $A$ -invariant subspace of  $\mathbb{C}^m$ . (15 points)

