Math 290 Name: Key Dr. Beezer
Exam 4 Spring 2021
Chapter VS

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Determine if the following subset of M_{22}, the vector space of 2×2 matrices, is linearly independent. (15 points)

$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix}, \begin{bmatrix} -4 & 1 \\ -7 & 8 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix} \right\}$$

Begin with a RLD:

$$a_1 \begin{bmatrix} 1 & 0 \\ 3 & -4 \end{bmatrix} + a_2 \begin{bmatrix} -4 & 1 \\ -7 & 8 \end{bmatrix} + a_3 \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} a_1 - 4a_2 + 3a_3 & a_2 - a_3 \\ 3a_1 - 7a_2 + 5a_3 & -4a_1 + 8a_2 - 2a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{matrix}]{\text{system w/ augmented}} \left[\begin{array}{ccc|c} 1 & -4 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & -7 & 5 & 0 \\ -4 & 8 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the <u>only</u> solution is $a=b=c=0$, which means that C <u>is</u> a linearly independent set.

2. Row-reduce the following matrix with Sage. Based on this computation, and with no further computations at all (by hand, nor by Sage), specify the rank and nullity of the matrix. Then determine, with explanation, the nicest possible basis for the column space of A, $\mathcal{N}(A)$. (15 points)

$$A = \begin{bmatrix} 1 & -1 & -7 & -2 & -2 \\ 0 & 1 & 2 & -1 & -4 \\ 0 & 2 & 4 & -1 & -6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -5 & 0 & 0 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

nullity $= \dim(\mathcal{N}(A)) =$ # free variables in homogeneous system $=$ # non-pivot columns $= 2$

rank + nullity = # columns

rank + 2 = 5 $\Rightarrow$ rank = 3 $\Rightarrow \dim(\mathcal{C}(A)) = 3$

A 3-dimensional subspace of $\mathbb{C}^3$ must equal $\mathbb{C}^3$ (Theorem EDYES)

The best basis is $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

3. Illustrate the use of Theorem TSS to determine that W is a subspace of the vector space $\mathbb{C}^2$. (20 points)

$$W = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| \, a + 2b = 0 \right\}$$

① $W \neq \emptyset$ $\quad 0+2(0)=0 \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$

② Assume $\underset{\sim}{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in W$ and $\underset{\sim}{y} = \begin{bmatrix} c \\ d \end{bmatrix} \in W$. So $a+2b=0$ & $c+2d=0$.

$\underset{\sim}{x}+\underset{\sim}{y} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$. check: $(a+c)+2(b+d)$ $\Rightarrow \underset{\sim}{x}+\underset{\sim}{y} \in W$

$= (a+2b)+(c+2d) = 0+0=0$

③ Assume $\underset{\sim}{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in W$ and $\alpha \in \mathbb{C}$. Know $a+2b=0$

$\alpha \underset{\sim}{x} = \begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}$. Check: $\alpha a + 2(\alpha b)$ $\Rightarrow \alpha \underset{\sim}{x} \in W$

$= \alpha(a+2b) = \alpha \cdot 0 = 0$

So, by Theorem TSS, W is a subspace of $\mathbb{C}^2$.

4. Find a basis for the following subspace of P_2, the vector space of polynomials with degree at most 2. (20 points)

$$X = \left\{ a + bx + cx^2 \,\middle|\, a - b + 5c = 0,\ 2a + b + c = 0 \right\}$$

The conditions are a homogeneous system of equations w/ coefficient matrix

$\begin{bmatrix} 1 & -1 & 5 \\ 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ So $a = -2c$ (c as the free variable)

$b = 3c$

Thus $X = \{-2c + (3c)x + cx^2 \mid c \in \mathbb{C}\} = \{c(-2+3x+x^2) \mid c \in \mathbb{C}\} = \langle \{-2+3x+x^2\} \rangle$

So $B = \{-2+3x+x^2\}$ spans X.

B is linearly independent, since it is a single nonzero vector.

So B is a basis of X.

5. Suppose that V is a vector space, and $\mathbf{v} \in V$. Prove that $0\mathbf{v} = \mathbf{0}$. (15 points)

This is Theorem ZSSM. See the proof there.

6. Suppose that $D = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_k\} \subseteq \mathbb{C}^m$ is a linearly independent set, and A is a nonsingular $m \times m$ matrix. Prove that $E = \{A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3, \ldots, A\mathbf{v}_k\} \subseteq \mathbb{C}^m$ is a linearly independent set. (15 points)

Begin with a RLD on E,

$$\mathbf{0} = a_1 A\mathbf{v}_1 + a_2 A\mathbf{v}_2 + \cdots + a_k A\mathbf{v}_k$$
$$= A(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k)$$

A nonsingular implies

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

D linearly independent implies

$$a_1 = a_2 = \cdots = a_k = 0$$

Which means E is linearly independent.