Math 290 Name: Key Dr. Beezer
Exam 1 Spring 2021
Chapter SLE

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices, except in the question that asks you to row-reduce without Sage. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Solve the following two systems of linear equations and express the solutions for each as a set of column vectors. (30 points)

(a)

$$\begin{aligned}-2x_1+2x_3+6x_4-2x_5&=7\\4x_1-3x_2+3x_3+3x_4+5x_5&=10\\-5x_1+4x_2-3x_3-x_4-5x_5&=-5\\-3x_1+x_2+x_3+5x_4-3x_5&=18\end{aligned}$$

Augmented matrix of the system

$$\left[\begin{array}{rrrrr|r}-2&0&2&6&-2&7\\4&-3&3&3&5&10\\-5&4&-3&-1&-5&-5\\-3&1&1&5&-3&18\end{array}\right]\xrightarrow[\text{(Sage)}]{\text{RREF}}\left[\begin{array}{rrrrr|r}1&0&0&0&2&0\\0&1&0&2&2&0\\0&0&1&3&1&0\\0&0&0&0&0&1\end{array}\right]$$

Last column is a pivot column so by Theorem RCLS there are no solutions. Solution set $=\emptyset=\{\}$

(b)

$$\begin{aligned}-2x_1+5x_2-8x_3&=5\\x_1-3x_2+5x_3&=-3\\x_1-x_2+2x_3&=0\\-x_1+4x_2-3x_3&=8\end{aligned}$$

Augmented matrix of the system

$$\left[\begin{array}{rrr|r}-2&5&-8&5\\1&-3&5&-3\\1&-1&2&0\\-1&4&-3&8\end{array}\right]\xrightarrow[\text{(Sage)}]{\text{RREF}}\left[\begin{array}{rrr|r}1&0&0&1\\0&1&0&3\\0&0&1&1\\0&0&0&0\end{array}\right]$$

Last column not a pivot column, so RCLS $\Rightarrow$ consistent system

$r=3=n$, CSRN $\Rightarrow$ unique solution

Solution set $=\left\{\begin{bmatrix}1\\3\\1\end{bmatrix}\right\}$

2. Without using Sage, find a matrix B in reduced row-echelon form which is row-equivalent to A. It is especially important to show all of your work, so it is clear you have not used Sage. (15 points)

$$A=\begin{bmatrix}1&-1&3&3\\2&-1&5&6\\2&0&4&6\end{bmatrix}\xrightarrow[-2R_1+R_3]{-2R_1+R_2}\begin{bmatrix}1&-1&3&3\\0&1&-1&0\\0&2&-2&0\end{bmatrix}\xrightarrow[-2R_2+R_3]{1R_2+R_1}\begin{bmatrix}1&0&2&3\\0&1&-1&0\\0&0&0&0\end{bmatrix}=B$$

3. Is the matrix B singular or not? Provide a justification for your answer. (15 points)

$$B = \begin{bmatrix} -5 & 2 & -3 & 7 & 2 \\ -1 & 2 & 2 & 7 & 2 \\ -2 & 1 & -1 & 3 & 1 \\ 1 & 1 & 3 & 5 & 1 \\ -3 & 3 & -2 & -1 & 3 \end{bmatrix} \xrightarrow[\text{(Sage)}]{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \neq I_5$$

So by Theorem NMRRI, B is <u>not</u> nonsingular,
i.e. B is singular.

4. Compute the null space of the matrix C, $\mathcal{N}(C)$. (15 points)

$$C = \begin{bmatrix} 1 & 2 & 8 & -8 & -2 \\ 0 & 1 & 2 & -2 & -3 \\ 0 & -2 & -4 & 4 & 7 \end{bmatrix} \xrightarrow[\text{(Sage)}]{\text{RREF}} \begin{bmatrix} 1 & 0 & 4 & -4 & 0 \\ 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So we want to describe the solution set for the homogeneous system $LS(C, 0)$.
x_1, x_2 & x_5 are dependent (see pivot columns) & x_3 & x_4 are free.
Equations rearranged

$$x_1 = -4x_3 + 4x_4$$
$$x_2 = -2x_3 + 2x_4$$
$$x_5 = 0$$

$$S = \left\{ \begin{bmatrix} -4x_3+4x_4 \\ -2x_3+2x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} \middle| \; x_3, x_4 \in \mathbb{C} \right\} = \mathcal{N}(C)$$

5. Each statement below is true or false. State which and give a **justification**. Answers without justifications will receive no credit. A good justification for "True" might be a theorem, and a good justification for "False" might be a simple counterexample (an example which demonstrates that the statement is not always true). (25 points)

(a) A homogeneous system is consistent.

True by Theorem HSC.

(b) A system with a nonsingular coefficient matrix has the zero vector as a solution.

$LS\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right)$ does not have the zero vector as a solution

↑ nonsingular by Theorem NMRRI. So False.

(c) A consistent system with 12 variables and 8 equations has infinitely many solutions.

True by Theorm CMVEI.

(d) A system with a singular coefficient matrix has no solutions.

$LS\left(\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right)$ has solution $x_1 = x_2 = 1$.

↑ singular, row-reduces to have a zero row.

So False.

(e) A system with 3 variables and 2 equations has infinitely many solutions.

$$2x_1 + x_2 + x_3 = 2$$
$$2x_1 + x_2 + x_3 = 832$$

No values of x_1 & x_2 & x_3 can make both equations true simultaneously. False.