Math 491 Friday, March 27 Chapter 21 Fields

- Sage 20 Monday
- Exam 19/20 Wednesday 8:30 Pacific
- Project Proposal Sunday Nite April 5 (Monday)
- Midterm Grades

F - field, E - field, $F \subseteq E$

E is an <u>extension</u> of F (E is a "superfield" of F)

<u>Ex</u> $\mathbb{R}$ field, $\mathbb{C}$ extension (obtain a root i of $x^2+1=0$)

<u>Ex</u> $\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$ extension of $\mathbb{Q}$

(obtain a root of $\underline{x^2-3}=0$, $\sqrt{3} \notin \mathbb{Q}$)

in $\mathbb{Q}[x]$

Ex $\mathbb{Q}(\sqrt{3}+\sqrt{7})$ Ex $\mathbb{Q}(\cos(\pi/24))$

$$\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \qquad (\pi ?)$$

We can build extension fields w/ one root of an irreducible polynomial

F - field, $p(x) \in F[x]$, $p(x)$ irreducible

$\Rightarrow$ $\langle p(x) \rangle$ maximal ideal $\Rightarrow$ $\dfrac{F[x]}{\langle p(x)\rangle}$ is a (quotient ring) field.

$\Rightarrow \bar{x} = \underline{\underline{x + \langle p(x)\rangle}} \in \dfrac{F[x]}{\langle p(x)\rangle}$ & $p(\bar{x}) = \bar{0}$

$\Rightarrow \dfrac{F[x]}{\langle p(x)\rangle}$ contains a copy of F $\left(= \{ f + \langle p(x)\rangle \mid f \in F \}\right)$

So $E = \dfrac{F[x]}{\langle p(x)\rangle}$ is an extension of F (w/ a root of $p(x)$)

Ex $\dfrac{\mathbb{R}[x]}{\langle x^2+1\rangle} \cong \mathbb{C}$ Ex $\dfrac{\mathbb{Q}[x]}{\underline{\langle x^2-3\rangle}} \cong \mathbb{Q}(\sqrt{3})$ Ex $GF(5^3)$

$\mathbb{Q}(-\sqrt{3})$

Sage: FiniteField(p^n, modulus = polynomial)
$\ast$ $\ast\ast$

Sage: NumberField(polynomial) $\longrightarrow$ $\mathbb{Q}$[root of polynomial]

Algebraic Elements

F - field α is algebraic element over F if α root of some (any) polynomial in $F[x]$.

Ex $\sqrt{3}$ algebraic over $\mathbb{Q}$
$x^2 - 3$

Ex $5/3$ algebraic over $\mathbb{Q}$
$x - 5/3$

Ex π is not algebraic over $\mathbb{Q}$, we say π is transcendental.

Defn E is an extension of F, then E is an algebraic extension if each element of E is algebraic over F.

Minimal polynomial of an element

Ex $\mathbb{Q}[(3+\sqrt{2})^{1/3}]$ $\alpha=(3+\sqrt{2})^{1/3}$ algebraic element?

$\alpha^3 = 3+\sqrt{2}$ $\sqrt{2} = \alpha^3-3$ $2=(\alpha^3-3)^2$

$2 = \alpha^6 - 6\alpha^3+9$ $0 = \alpha^6-6\alpha^3+7$

So α is a root of $p(x) = x^6 - 6x^3+7$, hence algebraic over $\mathbb{Q}$
$\in \mathbb{Q}[x]$

Turns out x^6-6x^3+7 is the minimal polynomial for $\alpha=(3+\sqrt{2})^{1/3}$

monic ✓, least degree w/ α as root ✓
$\Rightarrow$ irreducible

Sage QQ[(3+sqrt(2))**(1/3)]
generator a
a.minimum_polynomial( )

(a^2+1).minimum_polynomial( )

HW Cam Scanner
Genius Scan
Tiny Scanner