Math 491  Friday, March 27  Chapter 21 Fields

- Sage 20  Monday
- Exam 19/20  Wednesday  8:30 Pacific
- Project Proposal Sunday Nite  April 5 (Monday)
- Midterm Grades

\[ F - \text{field}, \ E - \text{field} \quad F \leq E \]
\[ E \text{ is an extension of } F \quad (E \text{ is a "super-field" of } F) \]
\[ E \in \mathbb{R} \text{ field, } C \text{ extension} \quad (\text{obtain a root } i \text{ of } x^2 + 1 = 0) \]
\[ E \in \mathbb{Q}(\sqrt{3}) = \left\{ a + b\sqrt{3} \mid a, b \in \mathbb{Q} \right\} \text{ extension of } \mathbb{Q} \]
\[ (\text{obtain a root of } x^2 - 3 = 0 \quad \text{in } \mathbb{Q}(\sqrt{7}) ) \]
\[ E \in \mathbb{Q}(\sqrt{17}) \quad E \in \mathbb{Q}(\cos(\pi/24)) \]

\[ \mathbb{N} \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{R} \quad (??) \]

We can build extension fields \( E \) over one root of an irreducible polynomial \( F \) - field, \( \mathbb{P}(\mathcal{X}) \in \mathbb{F}[\mathcal{X}] \), \( \mathbb{P}(\mathcal{X}) \) irreducible

\[ \Rightarrow \langle \mathbb{P}(\mathcal{X}) \rangle \text{ maximal ideal} \Rightarrow \frac{\mathbb{F}[\mathcal{X}]}{\langle \mathbb{P}(\mathcal{X}) \rangle} \text{ is a (quotient ring) field} \]

\[ \Rightarrow \bar{x} = \frac{\mathbb{F}[\mathcal{X}]}{\langle \mathbb{P}(\mathcal{X}) \rangle} \quad \mathbb{P}(\bar{x}) = \bar{0} \]

\[ \Rightarrow \frac{\mathbb{F}[\mathcal{X}]}{\langle \mathbb{P}(\mathcal{X}) \rangle} \text{ contains a copy of } \mathbb{F} \quad (= \{ f+\langle \mathbb{P}(\mathcal{X}) \rangle \mid f \in \mathbb{F} \}) \]

So \( E = \frac{\mathbb{F}[\mathcal{X}]}{\langle \mathbb{P}(\mathcal{X}) \rangle} \) is an extension of \( \mathbb{F} \) (\( \bar{x} \) a root of \( \mathbb{P}(\mathcal{X}) \))

\[ \bar{x} \in \frac{\mathbb{F}[\mathcal{X}]}{\langle \mathbb{P}(\mathcal{X}) \rangle} \quad \bar{x} \in \frac{\mathbb{F}[\mathcal{X}]}{\langle \bar{x}^2 - 3 \rangle} \quad \Rightarrow \mathbb{Q}(\sqrt{3}) \quad E \in \mathbb{Q}(\sqrt{3}) \]

\[ \text{Ex} \quad \mathbb{F}(2^{3}) \]

\[ \text{Ex} \quad \mathbb{Q}(\sqrt{-13}) \]
Sage: Finite Field ( \( p^n \), modulus = \( p \) polynomial)
Sage: Number Field ( polynomial) \( \rightarrow \mathbb{Q}[\text{root of polynomial}] \)

Algebraic Elements

\( F \)-field \( x \) is algebraic over \( F \) if \( x \) root of some (any) polynomial in \( F[x] \).

\( \sqrt{3} \) algebraic over \( \mathbb{Q} \) \( \sqrt{3} \) algebraic over \( \mathbb{Q} \)
\( x^2 - 3 \)

\( \pi \) is not algebraic over \( \mathbb{Q} \), we say \( \pi \) is transcendental.

Defn \( E \) is an extension of \( F \), then \( E \) is an algebraic extension
if each element of \( E \) is algebraic over \( F \).
Minimal polynomial of an element
\[ \mathbb{Q}[\sqrt[3]{3 + \sqrt{2}}] \]
\[ \alpha = (3 + \sqrt{2})^{\frac{1}{3}} \]

Algebraic element?

\[ \alpha^3 = 3 + \sqrt{2} \]
\[ \sqrt{2} = \alpha^3 - 3 \]
\[ 2 = (\alpha^3 - 3)^2 \]
\[ 2 = \alpha^6 - 6\alpha^3 + 9 \]
\[ 0 = \alpha^6 - 6\alpha^3 + 7 \]

So \( \alpha \) is a root of \( p(x) = x^6 - 6x^3 + 7 \), hence algebraic over \( \mathbb{Q} \)

Turns out \( x^6 - 6x^3 + 7 \) is the minimal polynomial for \( \alpha = (3 + \sqrt{2})^{\frac{1}{3}} \)

\[ \mathbb{Q} \subseteq \mathbb{Q}[\sqrt[3]{3 + \sqrt{2}}] \]

Sage
\[ \mathbb{Q} \subseteq \mathbb{Q}[\sqrt[3]{3 + \sqrt{2}}] \times \mathbb{Q}[\sqrt[3]{3 + \sqrt{2}}] \]

Generator \( \alpha \)

\( \alpha \), minimum polynomial()

\((\alpha^2 + 1)\), minimum polynomial()