An Exploration of the Normed Division Algebras

Riley Potts

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1 Introduction

The first division algebra is the real numbers, represented by $\mathbb{R}$. In the mid 16th century, Gerolamo Cardano is credited with being the first to introduce the complex i into into Mathematics in his attempts to find solutions to cubic equations.[1] As it happens, the result is an algebra which can be added, subtracted, multiplied, and divided in a similar manner to $\mathbb{R}$. We call this new two-dimensional division algebra $\mathbb{C}$, the complex numbers. Since $\mathbb{C}$ has been understood to be more useful, and therefore more common in mathematics, algebraists have been wondering if there exist any other constructions of numbers which can be added, subtracted, multiplied, and divided.[2] Surprisingly there exist exactly 4 objects which qualify as Normed Division Algebras: $\mathbb{R}$ the one-dimensional Reals, $\mathbb{C}$ the two-dimensional Complexes, $\mathbb{H}$ the four-dimensional Quaternions, and $\mathbb{O}$ eight-dimensional Octonions.[3]

2 What is a Division Algebra?

A Normed Division Algebra A is an algebra over a field, for our purposes $\mathbb{R}$ will be our base field, which is also a Division Ring and for which there exists a norm which satisfies:[4]

$$||ab|| \leq ||a||||b||$$

[1] Merino, Orlando. University of Rhode Island, Jan. 2006, www.math.uri.edu/ merino/spring06/mth562/ShortHistoryComplexNumbers2006.pdf.

[2] Wolchover, Natalie, and Quanta Magazine. “The Octonion Math That Could Underpin Physics.” Quanta Magazine, www.quantamagazine.org/the-octonion-math-that-could-underpin-physics-20180720/.

[3] Furey, Cohl. “Division Algebras and the Standard Model.” Division Algebras and the Standard Model. 21 Apr. 2020, Cambridge, England, https://www.youtube.com/channel/UCvsmxUuD5ZdOGittaeosXMA

[4] “NLab Normed Division Algebra.” Normed Division Algebra in NLab, ncatlab.org/nlab/show/normed division algebra

where the norm of an element $a \in A$ is defined as $\sqrt{aa^*}$ with $a^* \in A$ as the conjugate of a. A Division Ring is a non-empty set A together with the operations of addition and multiplication which satisfy the following properties[5]:

1. Additive Commutativity: $a + b = b + a,$ for all $a, b \in A$
2. Additive Associativity: $a + (b + c) = (a + b) + c,$ for all $a, b, c \in A$
3. Additive Identity: there exists $0 \in A$ such that $a+0 = 0+a = a,$ for all $a \in A$
4. Additive Inverse: for all $a \in A,$ there exists $-a \in A$ such that $a + (-a) = (-a) + a = 0$
5. Multiplicative Associativity: $a(bc) = (ab)c$ for all $a, b, c \in A$
6. Left and Right Distributivity: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in A$
7. Multiplicative Unity: there exists $1 \in A$ such that $a1 = 1a = a$ for all $a \in A$
8. All non-zero elements are units: for all $a \in A$ such that $a \neq 0,$ there exists $a^{-1} \in A$ where $a(a^{-1}) = (a^{-1})a = 1$

One implication of all of these conditions is that any mathematical object which satisfies this definition will also not have any so-called zero divisors. This means that whenever $ab = 0$, either $a = 0$ or $b = 0$. It is worth noting that some people do not hold so tightly to the condition of multiplicative associativity. Sometimes algebraists either drop the requirement of multiplicative associativity, or they replace it with weaker forms of a similar concept. We will explore what happens when this is allowed later in this essay. At this point, we have addition and multiplication. With the existence of additive and multiplicative inverses, we now define subtraction between elements as

$$a - b = a + (-b)$$

$$a - (-b) = a + (-(-b)) = a + b$$

for all $a, b \in A$. Likewise, we define division between elements as

$$a/b = a(b^{-1})$$
$$a/(b^{-1}) = a((b^{-1})^{-1}) = ab$$

for all $a, b \in A$.

We now have a system of numbers which can be added, subtracted, multiplied, and divided. $\mathbb{R}$ is the classic example of a division algebra, but if we

[5] Judson, Thomas W. Abstract Algebra: Theory and Application. https://books.aimath.org/aata/aata.html

let $\{\mathbf{1}, \mathbf{i}\}$ be a basis for $\mathbb{C}$, then we have a two dimensional division ring with elements of $\mathbb{C}$ being linear combinations with scalars taken from the field $\mathbb{R}$. However, this is not the only way to think of constructing a division algebra of higher dimension than one. The Cayley-Dickson Construction is an intuitive way to construct consecutive normed division algebras.

3 Cayley-Dickson Procedure

Realizing that complex numbers can be thought of as a pair of real numbers, the Cayley-Dickson Procedure aims at constructing a sequence of normed division algebras by pairing up elements from one algebra to make a new one with twice the dimension of the previous one.[6] To start, we are given $\mathbb{R}$ which is a normed division algebra. We can pair two elements $a, b \in \mathbb{R}$ to get the complex number $(a, b) = a + ib$. We define the complex conjugate of z as $z^* = (a, b)^* = (a, -b)$. Addition of these numbers is done elementwise and we define multiplication of two complex numbers $z, w \in \mathbb{C}$ with $z = (a, b)$ and $w = (c, d)$ as

$$(a, b)(c, d) = (ac - bd, ad + bc).$$

Note that the additive inverse of any $z = (a, b) \in \mathbb{C}$ is $(-a, -b)$. Becuase in the complex numbers there exists a norm which is defined as $||z|| = (zz^*)^{1/2}$, we say that $\mathbb{C}$ is a normed division algebra. Also note that the multiplicative inverse of $z \in \mathbb{C}, z \neq 0$ is

$$z^{-1} = \frac{z^*}{||z||^2} = \frac{(a, -b)}{(a^2 + b^2, 0)}.$$

With this guarantee of an additive and multiplicative inverse we are able to subtract and multiply elements as well as add and multiply them, and thus we do indeed have a normed division algebra. With this algebra called $\mathbb{C}$, we can repeat the Cayley-Dickson construction using pairs of complex numbers rather than pairs of the reals. This yields a new division algebra called the Quaternions, which we represent with $\mathbb{H}$. Let suppose we took a pair of elements $w, z \in \mathbb{C}$. Then we can represent an element of $\mathbb{H}$ with the ordered pair (w, z). Just as with $\mathbb{C}$ each element of $\mathbb{H}$ has a conjugate. For some $q \in \mathbb{H}$ when $q = (w, z)$, define the conjugate q^* to be

$$q^* = (w, z)^* = (w^*, -z)$$

Note that when $w = (a, b)$ and $z = (c, d)$:

$$q^* = ((a, -b), (-c, -d)).$$

As with the complexes, addition is done elementwise (as is subtraction). For two elements $p, q \in \mathbb{H}$ with $p = (w, z)$ and $q = (x, y)$, we define the multiplication of p and q as

[6] Baez, John. "The Cayley-Dickson Construction." The Cayley-Dickson Construction, math.ucr.edu/home/baez/octonions/node5.html.

$$pq = (w, z)(x, y) = (wx - bd^*, zx^* + yw).$$

There does also exist a norm for elements of $\mathbb{H}$ which is defined as follows for an element $p = (w, z) \in \mathbb{H}$:

$$\begin{aligned} ||p|| &= (pp^*)^{1/2} \\ &= ((w, z)(w^*, -z))^{1/2} \\ &= (ww^* + zz^*, 0)^{1/2} \\ &= (||w||^2 + ||z||^2, 0)^{1/2}. \end{aligned}$$

As with $\mathbb{C}$ the additive inverse of $p \in \mathbb{H}$ is simply $-p$, and the multiplicative inverse is

$$p^{-1} = \frac{p^*}{||p||^2}.$$

With the multiplicative and additive inverses we are now able to add, subtract, multiply, and divide elements of $\mathbb{H}$, making $\mathbb{H}$ a division algebra which happens to have dimension 4, twice the dimension of $\mathbb{C}$ Having successfully applied the Cauchy-Dickson method of construction to the create both $\mathbb{C}$ and $\mathbb{H}$, it should not come as surprising that we can do this again to construct another normed division algebra. This next algebra, which double in dimension to eight, is represented by $\mathbb{O}$ and called the octonions. Suppose we have two quaternions p, q. Then we can represent some $f \in \mathbb{O}$ with $f = (p, q)$. With $\mathbb{O}$, we define addition, multiplication, subtraction, and division as well as the additive inverse, multiplicative inverse, conjugate, and norm in exactly the same way as we defined them for the quaternions. Because of this, it is apparent that the octonions satisfy the definition of a normed division algebra. We are able to carry out the steps of the Cayley-Dickson construction if we wish to continue producing algebras, but curiously enough, each algebra we create after the octonions does not satisfy the definition of a division algebra, as we are not able to guarantee the existence of a multiplicative inverse for each element of the resulting algebras. In fact, the theorem stated in a paper titled *The Octonions* by John C. Baez, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are the only normed division algebras[7]. This means that aside from any algebra that the Cayley-Dickson method of construction produces, there are also no algebras of dimension three, five, six, or seven.

4 Modeling Division Algebras as Vector Spaces

As we saw with the Cayley-Dickson Construction method, elements of $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ can be modeled as an ordered pair of elements from the previous algebra. If we take $\{\mathbf{1}\}$ as a basis for the reals, we can see that if we let our vector scalars be elements of $\mathbb{R}$ we are able to model any real number as a "linear combination"

[7] Baez, John C. "The Octonions." Bulletin of the American Mathematical Society, vol. 39, no. 02, 2001, pp. 145–206., doi:10.1090/s0273-0979-01-00934-x.

of vectors from our basis. Suppose that we keep $\mathbb{R}$ as the set from which we grab scalars. Then it becomes clear that with $\{\mathbf{1}, \mathbf{i}\}$ as a basis for $\mathbb{C}$ we are able to model any $z = a + b\mathbf{i}$ in $\mathbb{C}$ as a linear combination of our basis vectors. A similar thing exists for both the quaternions and the octonions. Suppose we let $\{\mathbf{1}, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$. We define $\mathbf{I}, \mathbf{J}, \mathbf{K}$ so that

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = -\mathbf{1}$$

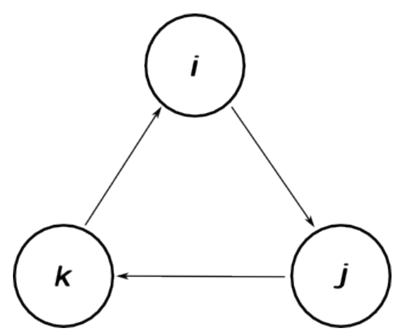
We can show that a multiplication which satisfies our earlier construction's definition is as

$$\mathbf{IJ} = \mathbf{K}$$

$$\mathbf{JK} = \mathbf{I}$$

$$\mathbf{KI} = \mathbf{J}$$

However, it can also be shown that when the order of multiplication is reversed for the above equations, the result is the same, but negative instead. Here when one follows the arrows around the circle the resulting multiplication is positive, but when one moves around the circle against the arrows, the result is negative. With these rules, all elements of the quaternions can be made as a linear combinations of the basis elements $\{\mathbf{1}, \mathbf{I}, \mathbf{J}, \mathbf{K}\}$ with $\mathbb{R}$ as the set from which our scalars will come. Thus, any quaternion q can be written as $q = a\mathbf{1} + b\mathbf{I} + c\mathbf{J} + d\mathbf{K}$ where $a, b, c, d \in \mathbb{R}$. Note that by applying the previous definition of the conjugate of a quaternion to the representation of a quaternions as a linear combination, the conjugate of $q = a + b\mathbf{I} + c\mathbf{J} + d\mathbf{K} \in \mathbb{H}$ is given by



$$q^* = a - b\mathbf{I} - c\mathbf{J} - d\mathbf{K}$$

5 Properties of Division Algebras

Another curiosity that arises from the construction of these consecutive algebras is that as we increase the dimension of the algebra we are working with, we end up progressively losing some of the properties about the real numbers which are most familiar to us. Indeed, here is when it becomes helpful to be flexible in how essential multiplicative associativity is to your definition of a division algebra. Our archetypal division algebra is of course the real numbers. They behave quite nicely as they are both ordered and multiplicatively commutative. Moving onto the two dimensional complex numbers, we lose the ablitiy to order elements. Since it is impossible to say whether or not $(1, 0)$ is larger than $(0, 1)$, the elements of $\mathbb{C}$ are unordered. One interesting result of this fact is that unlike the reals, there only exists one infinity. In the reals, ever increasing numbers tend to positive infinity and ever decreasing numbers tend to negative infinity.

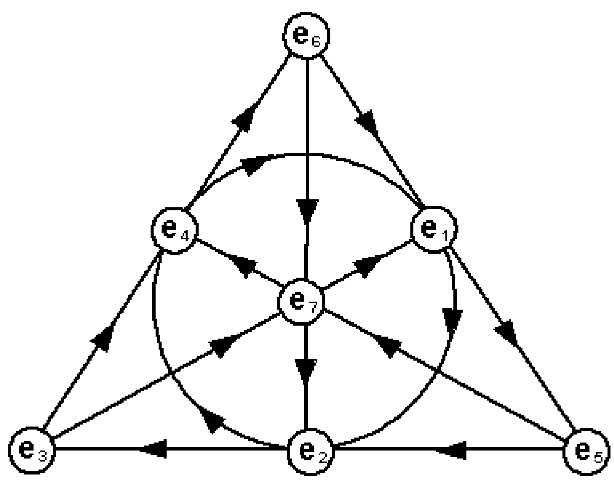
in the complexes, there is only one infinity, so for any $z = (a, b)$, z tends towards ∞ as either a or b tend toward $\pm\infty$. Next, while multiplication of quaternions can be a somewhat tedious affair, one can take two arbitrary quaternions and show that for any $p, q \in \mathbb{H}$ such that $p \neq \pm 1, q \neq \pm 1, \text{and} p \neq q$. Then

$$pq = -qp.$$

With this , we can see that once we move from $\mathbb{C}$ to $\mathbb{H}$ we lose another property which is familiar to us, as the quaternions do not retain the property of multiplicative commutativity which we have come to expect from algebra. Because of this, much of the mathematical computation of the quaternions tends to be done using some sort of computational software, rather than by hand.
Moving to our next and last normed division algebra, we have the octonions which can also be represented as an 8 dimensional vector space with scalars from $\mathbb{R}$ and a basis for $\mathbb{O}$ as $\{\mathbf{1}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}, \mathbf{e_5}, \mathbf{e_6}, \mathbf{e_7}\}$. Thus, supposing $a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in \mathbb{R}$, we can represent an octonion $f \in \mathbb{O}$ as

$$f = a_0 + a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3} + a_4\mathbf{e_4} + a_5\mathbf{e_5} + a_6\mathbf{e_6} + a_7\mathbf{e_7}$$

Here, all $\mathbf{e_i} \in \mathbb{O}$ share the property that $\mathbf{e_i}^2 = -\mathbf{1}$. Additionally, the rules for multiplication between the various basis vectors of the octonions becomes so complicated so quickly, that the easiest way to demonstrate multiplicative relationship between the non-identity elements of our basis is with the following graph called a Fano Plane[8]:



Here, there are 7 line segments including the central circle. each line segment has three elements along it, and each element is connected to every other via some line segment. To multiply two elements, pick them on the map, and then follow them to the remaining element along the line segment that they share.

[8]Baker, Martin J. "Maths - Octonion." Maths - Octonion - Martin Baker, 2017, www.euclideanspace.com/maths/algebra/realNormedAlgebra/octonion/index.htm

Similar to the multiplication graph for the quaternions, if you move with the arrows to arrive at the resulting element, the product is positive and if you must move against the arrows to find the resulting element, then the result is negative. For example, suppose we wish to compute $\mathbf{e_1} * \mathbf{e_6}$. Starting on $\mathbf{e_1}$, we find the line segment which is shared with $\mathbf{e_6}$. moving in the direction toward $\mathbf{e_6}$, we continue forward and loop back around to the third element which shares the line segment with them. Since we moved with the arrows to arrive at this result, we compute that $\mathbf{e_1} * \mathbf{e_6} = \mathbf{e_4}$. We saw that after $\mathbb{R}$ each algebra loses the ability to be ordered, and after $\mathbb{C}$ each algebra loses the property of multiplicative commutativity. $\mathbb{H}$ is multiplicatively associative, but are the octonions? Let us compute:

$$(\mathbf{e_3} * \mathbf{e_4}) * \mathbf{e_2} = \mathbf{e_6} * \mathbf{e_2} = -\mathbf{e_7}.$$

Next we compute:

$$\mathbf{e_3} * (\mathbf{e_4} * \mathbf{e_2}) = \mathbf{e_3} * (-\mathbf{e_1}) = -(-\mathbf{e_7}) = \mathbf{e_7}.$$

Therefore, $(\mathbf{e_3} * \mathbf{e_4}) * \mathbf{e_2} \neq \mathbf{e_3} * (\mathbf{e_4} * \mathbf{e_2})$. Since we have found $f, g, h \in \mathbb{O}$ such that $(fg)h \neq f(gh)$, we have the result that $\mathbb{O}$ is not necessarily multiplicatively associative. However, even though $\mathbb{O}$ is not multiplicatively associative, it does satisfy a weaker form of associativity, namely $\mathbb{O}$ is called alternative. That is the octonions satisfy the conditions that for all $f, g \in \mathbb{O}$[9]

$$(ff)g = f(fg) \qquad (fg)f = f(gf) \qquad (gf)f = g(ff).$$

We now have found an algebra which follows $\mathbb{H}$ from the Cayley-Dickson method of construction that like $\mathbb{C}$ is not ordered, like $\mathbb{H}$ is not multiplicatively commutative, and is itself not multiplicatively associative. This fact has made many mathematicians and physicists uncomfortable since their discovery and as a result, they sat without much use for a long time. Being that the octonions are alternative, together with the fact that for each element there exists a multiplicative inverse is enough to make most algebraists reconsider the definition of a normed division algebra to necessarily include multiplicative associativity. Supposing we do indeed allow $\mathbb{O}$ into our definition of division algebras, and we count that we now have 4 objects, which satisfy the definition of a normed division algebra. When we repeat the Cayley-Dickson construction method again and make pairs of octonions, we arrive at an algebra called the sedenions, $\mathbb{S}$ which are neither multiplicatively commutative, associative, nor even alternative. Moreover, for all algebras of greater dimension than the octonions, there exist zero divisors and thus they do not meet the definition of normed division algebras.[10]

[9] Schafer, Richard Donald. An Introduction to Nonassociative Algebras. Academic Press, 1966.
[10] Ibid.

6 Applications

The applications of division algebras are far reaching and quite fundamental to much of our current scientific modeling of the universe. Indeed, we find use of the real numbers in every facet of every person's daily life. They are by far the most useful and most universally understood division algebra. Apart from the real numbers, the complex numbers have proven to be very useful in that they are the mathematical underpinning of quantum mechanics in physics. The complex numbers also provide some of the most rich and elegant pure mathematics to be found, this in the form of complex analysis.

The quaternions have gained more widespread use in the last century with the advent of Albert Einstein's theories of special and general relativity. The mathematics of relativity are built upon quaternionic algebra and are what physicists have been using to model and understand how gravitational force behaves and interacts with massive objects. In computing, quaternionic computation makes modeling and graphing rotations of objects in three dimensional space far easier than it would be otherwise. Rather than treating some $q \in \mathbb{H}$ where $q = a\mathbf{1} + b\mathbf{I} + c\mathbf{J} + d\mathbf{K}$ as some ordered quadruplet (a, b, c, d), we say that it is equal to $(a, \mathbf{V})$ where $\mathbf{V}$ is some position vector. We can then scale the our quaternion to give it a value which is a multiple of q, but instead has the property that $||q'|| = 1$. From here, computations of graphic rotation become much simpler to program and faster to execute.[11]

As Cohl Furey points out in her lecture on online lecture series *Division Algebras and the Standard Model*, for a long time, the octonions did not have much of a place in physics. She says that she found it unlikely that "nature would rely so heavily on the first three division algebras, yet completely ignore the fourth and final one."[12] Because of this, her area of research is in using the octonions as a way to not just describe the standard model of particle physics, but also to attempt to explain why the standard model works the way it does and relies on the mathematical structures it does. It is exciting see that algebra has such a big role in the forefront of research physics in a way that is so broadly deepening our understanding of how the fundamental forces of nature work. Many people, myself included, believe that study of the octonions may be our gateway forward into moving past the standard model to what lies behind the scenes, and into the next evolution of our physical understanding of our world.[13]

[11] "Lecture on Quaternions in Computing." Davis, California.

[12] Furey, Cohl."Division Algebras and the Standard Model." Division Al-gebras and the Standard Model.21 Apr.2020, Cambridge, England,https://www.youtube.com/channel/UCvsmxUuD5ZdOGittaeosXMA

[13] ibid