A Brief Exploration of Normed Division Algebras

From $\mathbb{R}$ to $\mathbb{O}$ (and beyond?)

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What is a Normed Division Algebra?

A Normed Division Algebra is a set, together with an additive operation and a multiplicative operation which satisfy a certain set of conditions, namely:

1. The Norm is "friendly", meaning that $||ab|| \leq ||a||||b||$
2. Additive Commutativity
3. Additive Associativity
4. Additive Identity
5. Additive Inverses
6. Left and Right Distributivity
7. Multiplicative Identity (or Unity)
8. All non-zero elements are Units (Multiplicative Inverses)
9. Multiplicative Associativity (Alternativity)

Alternativity and Power Associativity

- Alternative Algebras satisfy the condition that for all a, b
  - $a(ab) = (aa)b \qquad a(ba) = (ab)a \qquad b(aa) = (ba)a$
- Power Associative Algebras satisfy the condition that for consecutive multiplication on identical elements, the order of multiplication does not matter.
  - Ex: $x * (x * (x * x)) = (x * (x * x)) * x = (x * x) * (x * x)$

Subtraction and Division

Subtraction:

$$a - b = a + (-b)$$

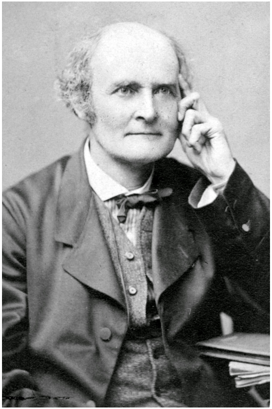
$$a - (-b) = a + (-(-b)) = a + b$$

Division:

$$\frac{a}{b} = a * (b^{-1})$$

$$\frac{a}{b^{-1}} = a * ((b^{-1})^{-1}) = a * b$$

Cayley and Dickson



Arthur Cayley

Leonard Eugene Dickson

Cayley-Dickson Procedure

William Rowan Hamilton was one of the first people to seriously treat the complex numbers as an ordered pair of real numbers, represented with

$$z = a + bi = (a, b)$$

The Cayley-Dickson Procedure aims at generalizing this concept as a way to create new algebras.

Cayley-Dickson Procedure: from $\mathbb{R}$ to $\mathbb{C}$

- Take $\mathbb{R}$ to be the base field. Then we can construct $\mathbb{C}$ by making ordered pairs of elements in $\mathbb{R}$, such as (a, b) where $a, b \in \mathbb{R}$.
- We define the conjugate of some $z \in \mathbb{C}$ as $z^* = (a, b)^* = (a, -b)$
- The Norm of some $z = (a, b)$ is defined as $||z|| = (zz^*)^{1/2}$

Cayley-Dickson Procedure: from $\mathbb{R}$ to $\mathbb{C}$

- The additive inverse of some $(a, b) \in \mathbb{C}$ is given by $-(a, b) = (-a, -b)$
- Addition and subtraction are computed elementwise
- For some $z = (a, b),\ w = (c, d)$ multiplication is defined as $zw = (a, b)(c, d) = (ac - bd, ad + bc)$
- The multiplicative inverse of $z = (a, b)$ is $z^{-1} = \frac{z^*}{||z||^2}$

The Game Continues: CDP from $\mathbb{C}$ to $\mathbb{H}$

We can repeat this process, using $\mathbb{C}$ as the base field. Let $z, w \in \mathbb{C}$:

- Elements of $\mathbb{H}$ can be represented as (z, w), where $z, w \in \mathbb{C}$
- The conjugate of some $(z, w) = q \in \mathbb{H}$ is given by $q^* = (z^*, -w)$
- The Norm of some $q = (z, w)$ is given by $||q|| = (qq^*)^{1/2}$

The Game Continues: CDP from $\mathbb{C}$ to $\mathbb{H}$

- The additive inverse of some $(z, w) \in \mathbb{H}$ is given by $-(z, w) = (-z, -w)$
- Addition and subtraction are computed elementwise
- For some $p = (z, w), q = (x, y) \in \mathbb{H}$, multipication is given by $pq = (z, w)(x, y) = (zx - yw^*, z^*x + xw)$
- The multiplicative inverse of some $q \in \mathbb{H}$ is given as $q^{-1} = \frac{q^*}{||q||^2}$

The Game Continues: CDP from $\mathbb{H}$ to $\mathbb{O}$

Again we repeat this process by pairing up elements of $\mathbb{H}$ to form octonions. We can represent any $f \in \mathbb{O}$ as $f = (p, q)$ for some $p, q \in \mathbb{H}$.

We define the Norm, conjugate, additive inverse, multiplicative inverse, addition, subtraction, multiplication, and division exactly the same as we did in $\mathbb{H}$.

The Game Continues: CDP from $\mathbb{O}$ to $\mathbb{S}$

We can continue the Cayley-Dickson procedure ad infinitum and find that just as with the octonions, there are no changes in definitions.

However, once we create the sedenions, $\mathbb{S}$, we find that we lose the ability to guarantee multiplicative inverses and start finding zero divisors.

Cayley-Dickson Algebra Properties

- $\mathbb{R}$: Ordered, multiplicatively commutative, multiplicatively associative, alternative, power associative
- $\mathbb{C}$: Multiplicatively commutative, multiplicatively associative, alternative, power associative
- $\mathbb{H}$: Multiplicatively associative, alternative, power associative
- $\mathbb{O}$: Alternative, power associative
- $\mathbb{S}$: Power associative

Octonion Multiplication

Suppose some octonion $f = (p, q)$ with $p, q \in \mathbb{H}$.

Then there exist some $x, y, w, z \in \mathbb{C}$ such that $p = (x, y)$ and $q = (w, z)$.

With this, there exist some $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \in \mathbb{R}$ such that $x = (a_1, a_2), y = (a_3, a_4), w = (a_5, a_6), z = (a_7, a_8)$.

We use these different representations to show that we can breakdown any octonion into its components which come from $\mathbb{R}$:

$$f = (p, q) = ((x, y), (w, z)) = (((a_1, a_2), (a_3, a_4)), ((a_5, a_6), (a_7, a_8)))$$

Octonion Multiplication

Let $\{\mathbf{1}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}, \mathbf{e_5}, \mathbf{e_6}, \mathbf{e_7}\}$ be a basis for $\mathbb{O}$. Then we can let our scalars come from $\mathbb{R}$ and represent any octonion as a linear combination of the basis vectors. We say that for some $f \in \mathbb{O}$

$$f = a_0 + a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3} + a_4\mathbf{e_4} + a_5\mathbf{e_5} + a_6\mathbf{e_6} + a_7\mathbf{e_7}$$

Multiplication of octonions becomes quite cumbersome when treated as ordered pairs, but it gets easier when each octonion is treated as a vector.

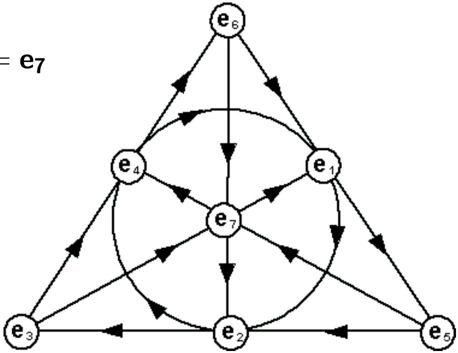
Octonion Multiplication

$(\mathbf{e}_3\mathbf{e}_4)\mathbf{e}_2 = \mathbf{e}_6\mathbf{e}_2 = -\mathbf{e}_7.$

$\mathbf{e}_3(\mathbf{e}_4\mathbf{e}_2) = \mathbf{e}_3(-\mathbf{e}_1) = -(-\mathbf{e}_7) = \mathbf{e}_7$

Therefore
$(\mathbf{e}_3\mathbf{e}_4)\mathbf{e}_2 \neq \mathbf{e}_3(\mathbf{e}_4\mathbf{e}_2)$

This Mnemonic is called the Fano plane and is use to remember the multiplication of basis vectors



Applications

- $\mathbb{R}$ is used everywhere, everyday, by everbody
- $\mathbb{C}$ is used in quantum physics
- $\mathbb{H}$ is used in the mathematics that underly relativity, as well as for modeling rotations in computer graphics
- Until very recently, $\mathbb{O}$ has not had much use for anything. Cohl Furey is currently attempting to use $\mathbb{O}$ to explain why the standard model of particle physics works the way that it does.