Introductory Multi-linear Algebra: From Dual Spaces to Tensor Products

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1 Introduction

In the same way that vector spaces embody linearity, Tensor Products embody multi-linearity. As such tensors (the elements of tensor products) are a widely applicable generalization of vectors and Vector Spaces.

In this paper, I seek to give an introductory, abstract understanding of tensors and Tensor Products. This paper assumes knowledge commensurate with an introductory undergraduate course in Linear Algebra and an undergraduate course in Abstract Algebra.

2 Dual Spaces

To understand Tensor Products it is easiest to begin with Dual Spaces and linearity.

Given finite dimensional vector spaces V and W over a field F we define the set of all linear transformations $Hom(V,W)$. [1]

Theorem 1. *Given two finite dimension vector spaces V and W over a field F, $Hom(V,W)$ forms a vector space over F where. for $S,T \in Hom(V,W), \alpha \in F$, and $\vec{v} \in V$, vector addition is defined: $(S+T)(\vec{v} = S(\vec{v}) + T(\vec{v})$ and scalar multiplication is defined: $(\alpha S)(\vec{v}) = \alpha S(\vec{v})$. [1]*

Proof. Let $\vec{v}, \vec{u} \in V, \alpha, \beta \in F$, and $\vec{w} \in W$.

Grab $S,T \in Hom(V,W)$. Then,

$$\begin{aligned}(S+T)(\vec{u}+\vec{v}) &= S(\vec{u}+\vec{v}) + T(\vec{u}+\vec{v}) \\ &= S(\vec{u}) + S(\vec{v}) + T(\vec{u}) + T(\vec{v}) \\ &= S(\vec{u}) + T(\vec{u}) + S(\vec{v}) + T(\vec{v}) \quad \text{Vector addition is commutative in } W \\ &= (S+T)(\vec{u}) + (S+T)(\vec{v})\end{aligned}$$

So, $(S+T) \in Hom(V,W)$ and addition is closed on $Hom(V,W)$.

And,

$$\begin{aligned}(S+T)(\vec{v}) &= S(\vec{v}) + T(\vec{v}) \\ &= T(\vec{v}) + S(\vec{v}) \qquad \text{Vector addition is commutative in } W \\ &= (T+S)(\vec{v})\end{aligned}$$

So, addition in $Hom(V,W)$ is commutative.

Grab $S,T,R \in Hom(V,W)$. Then,

$$\begin{aligned}((S+T)+R)(\vec{v}) &= (S+T)(\vec{v}) + R(\vec{v}) \\ &= S(\vec{v}) + T(\vec{v}) + R(\vec{v}) \\ &= S(\vec{v}) + (T+R)(\vec{v}) \\ &= (S+(T+R))(\vec{v})\end{aligned}$$

So, addition in $Hom(V,W)$ is associative.

Define $O : V \to W$ by $O(\vec{v}) = \vec{0}$. $O \in Hom(V,W)$. Grab $S \in Hom(V,W)$. Then, $(S+O)(\vec{v}) = S(\vec{v})+O(\vec{v}) = S(\vec{v})+\vec{0} = S(\vec{v})$. Addition is commutative, so $(O+S)\vec{v} = S(\vec{v})$ So, $Hom(V,W)$ is non-empty and contains an identity element.

Let $S(\vec{v}) = \vec{w}$. Define $\hat{S} : V \to W \in Hom(V,W)$ by $\hat{S(\vec{v})} = -\vec{w}$. Then,

$$\begin{aligned}(S + \hat{S})(\vec{v}) &= S(\vec{v}) + \hat{S(\vec{v})} \\ &= \vec{w} + (-\vec{w}) \\ &= \vec{0} \\ &= O(\vec{v})\end{aligned}$$

So, $\hat{S} = S^{-1}$ and for each $S \in Hom(V,W)$ there exists a $S^{-1} \in Hom(V,W)$.

Grab $S \in Hom(V,W)$. Then,

$$\begin{aligned}(\alpha S)(\vec{u} + \vec{v}) &= \alpha(S(\vec{u} + \vec{v}) \\ &= \alpha(S(\vec{u} + S(\vec{v}) \\ &= (\alpha S)(\vec{u}) + (\alpha S)(\vec{v})) \quad \text{Scalar multiplication is distributive in } W\end{aligned}$$

So, $\alpha(S) \in Hom(V,W)$ and scalar multiplication is closed.
Then consider $\alpha(\beta S)$,
$(\alpha(\beta S(\vec{v})) = \alpha S(\beta\vec{v}) = S(\alpha\beta\vec{v}) = \alpha\beta S(\vec{v}) = (\alpha\beta S)(\vec{v})$.
So, scalar multiplication is associative in $Hom(V,W)$.

Then consider $\alpha(S + T)$. $(S + T)(\vec{v}) = S(\vec{v}) + T(\vec{v})$. Note that $S(\vec{v})$ and $T(\vec{v})$ are elements of W. So, $(\alpha(S+T)(\vec{v}) = \alpha(S(\vec{v})+T(\vec{v})) = \alpha(S(\vec{v})+\alpha(T(\vec{v})$.
So, scalar multiplication distributes across vector addition in $Hom(V,W)$.

Then consider $(\alpha + \beta)S$,

$$\begin{aligned}((\alpha + \beta)S)(\vec{v}) &= (\alpha + \beta)(S\vec{v}) \\ &= \alpha S(\vec{v}) + \beta S(\vec{v}) \\ &= (\alpha S)(\vec{v}) + (\beta S)(\vec{v})\end{aligned}$$

So, scalar multiplication is distributive across scalar addition in $Hom(V,W)$.

Finally consider $1S$.

$$\begin{aligned}(1S)(\vec{v}) &= 1S(\vec{v}) \\ &= S(\vec{v}) \qquad \text{Definition of 1 in } W\end{aligned}$$

So, $Hom(V,W)$ has multiplicative identity.
Therefore, $Hom(V,W)$ fulfills the axioms which describe a vector space and $Hom(V,W)$ is a vector space for our defined operations. $\square$

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Given a F-vector space V, we define the dual space of V to be $V^* = Hom(V, F)$. The elements of V^* are linear forms, linear functional, or linear maps. Then, V^* is a vector space over F.

Theorem 2. *Let $B = \vec{v_1}, \dots, \vec{v_n}$ be a basis of V. For $\vec{v} \in V$ where $\vec{v} = a_1\vec{v_1} + \dots + a_n\vec{v_n}$ define $\phi_i : V \to F \in Hom(V, F)$ by $\phi_i(\vec{v}) = a_i$. Then, $B^* = \{\phi_1, \dots, \phi_n\}$ forms a basis of V^*. B^* is a dual basis.*

Proof. Let $\sum_{i=1}^{n} \rho_i(\phi_i) = id$ and $\vec{v_k} \in B$.

Then,

$$\begin{aligned}
\sum_{i=1}^{n} \rho(\phi_i(\vec{v_k})) &= id(\vec{v_k}) \\
\rho_1(\phi_1(\vec{v_k})) + \dots + \rho_k(\phi_k(\vec{k})) + \dots + \rho_n(\phi_n(\vec{v_k})) &= \vec{0} \\
\rho_1(\vec{0}) + \dots + \rho_k(\vec{1}) + \dots + \rho_n(\vec{0}) &= \vec{0} \\
\rho_k &= 0
\end{aligned}$$

So, for each basis vector $\vec{v_k}, \rho_k = 0$. B spans V. V is the domain for V^*, so B^* is linearly independent.

Grab $S \in V^*$ and $\vec{v} \in V$. $\vec{v} = a_1\vec{v_1} + \dots + a_n\vec{v_n}$.

First note, because S is linear, $S(a\vec{v}) = aS(\vec{v})$.

Consider,

$$\begin{aligned}
S(\vec{v}) &= S(a_1\vec{v_1} + \dots + a_n\vec{v_n}) \\
&= S(a_1\vec{v_1}) + \dots + S(a_n\vec{v_n}) \\
&= a_1S(\vec{v_1}) + \dots + a_nS(\vec{v_n}) \\
&= \phi_1(\vec{v_1})S(\vec{v_1}) + \dots + \phi_n(\vec{v_n})S(\vec{v_n}) \\
&= \phi_1(\vec{v_1})f_1 + \dots + \phi_n(\vec{v_n})f_n && \text{where } S(\vec{v_i}) = f_i \text{ and } f_i \in F \\
&= \phi_1(\vec{v})f_1 + \dots + \phi_n(\vec{v})f_n && \text{Note that } \phi_i(\vec{v_i}) = \phi_i(\vec{v})
\end{aligned}$$

So, B^* spans V^* and B^* forms a basis of V^*. ☐

The following corollary is then obvious but important:

Corollary 1 (Theorem 2). *Given a finite vector space V, $V \simeq V^*$.*

Proof. $dim(V) = |B| = n = |B^*| = dim(V^*)$. So, $dim(V) = dim(V^*)$. Vector spaces of the same dimension are isomorphic. ☐

We can also consider the more general case, $Hom(V, W)$, where V and W are vector spaces over field F.

Corollary 2 (Theorem 2). *Given finite vector spaces V and W over field F with $dim(V) = n$ and $dim(W) = m, dim(Hom(V, W) = nm$*

Proof. Let a_i, b_i, c_i be scalars in F, $\{\vec{v_1}, \ldots, \vec{v_n}\}$ be a basis of V and $\{\vec{w_1}, \ldots, \vec{w_m}\}$ be a basis of W.

Consider the set $B = \{\phi_{kl} \in Hom(V, W) : \phi_{kl}(a_1\vec{v_1} + \cdots + a_k\vec{v_k} + \cdots + a_n\vec{v_n}) = \vec{w_l}\}$

Let $\sum_{k=1}^{n}\sum_{l=1}^{m} a_{kl}\phi_{kl} = id$. Then,

$$\begin{aligned}
(\sum_{k=1}^{n}\sum_{l=1}^{m} a_{kl}\phi_{kl})(\vec{v_k}) &= O(\vec{v_k}) \\
\sum_{k=1}^{n}\sum_{l=1}^{m}(a_{kl}(\phi_{kl}(\vec{v_k}))) &= O(\vec{v_k}) \\
\sum_{l=1}^{m} a_{kl}\vec{w_l} &= \vec{0} \\
a_{kl} &= 0 \qquad\qquad \vec{w_i} \text{ are linearly independent}
\end{aligned}$$

So, for each, $v_k \in \{\vec{v_1}, \ldots, \vec{v_n}\}, a_{kj} = 0$ for $1 \leq j \leq m$.

We can then conclude that B is linearly independent.

Without loss of generality, consider $S \in Hom(V, W)$ where $S(\vec{v}) = \vec{w}$. Then,

$$\begin{aligned}
S(\vec{v}) &= S(a_1\vec{v_1} + \cdots + a_n\vec{v_n}) \\
&= a_1S(\vec{v_1}) + \cdots + a_nS(\vec{v_n}) \\
&= a_1(b_1\vec{w_1} + \ldots b_m\vec{w_m}) + \ldots a_n(c_1\vec{w_1} + \ldots c_m\vec{w_m}) \\
&= a_1(b_1\phi_{11}(\vec{v}) + \ldots b_m\phi_{1m}(\vec{v}) + \cdots + a_n(c_1\phi_{n1}(\vec{v}) + \ldots c_m\phi_{nm}(\vec{v}) \\
&= d_{11}\phi_{11}(\vec{v}) + \cdots + d_{nm}\phi(\vec{v})
\end{aligned}$$

So, any $S \in Hom(V, W)$ can be expressed as a linear combination of $\phi_i \in B$. So, B spans $Hom(V, W)$.

Then, B forms a basis of $Hom(V, W)$. So $dim(Hom(V, W)) = |B| = nm$. □

Proposition 1. *First, for all $\vec{v} \in V, \vec{v} \neq \vec{0}, \exists\varphi \in V^*$ such that $\varphi(\vec{v}) = 0$. Second, $\vec{v} = \vec{0}$ if and only if $\phi(\vec{v}) = 0$ for all $\phi \in V^*$. [4]*

Proof. The zero transformation is an element of V^* and for all $\vec{v} \in V$ where $\vec{v} \neq \vec{0}, id(\vec{v}) \neq 0$.

First, grab $\vec{0} \in V$ and $S \in V^*$. Then, $S(\vec{0}) = 0S(\vec{v}) = \vec{0}$. Then, let $\vec{v} \in V$ and assume $\phi(\vec{v}) = 0 \forall\phi \in V^*$. Consider, $id \in V^*$.

If $id(\vec{v}) = 0$. Obviously, $\vec{v} = \vec{0}$. □

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We can also consider the dual space of the dual space. That is the double (algebraic) dual space $(V^*)^* = V^{**}$ which consists of all linear forms $\omega : V^* \to F$. So, the elements of V^{**} are maps which assign scalar values to each linear form in V^*. [2]

Remark 1. *V^{**} is a vector space over F.*

It is then natural to consider the map $\bar{\omega}_{\vec{v}} : V^* \to F$ defined by $\omega_{\vec{v}}(S) = S(\vec{v})$. This map is obviously an evaluation homomorphism.

Proposition 2. $\omega_{\vec{v}} \in V^{**}$

Proof. let $S, T \in V^*$ and $a, b \in F$. Then,

$$\begin{aligned}\omega_{\vec{v}}(aS + bT) &= (aS + bT)(\vec{v}) \\ &= aS(\vec{v}) + bT(\vec{v}) = a\omega_{\vec{v}}(S) + b\omega_{\vec{v}}(T)\end{aligned}$$

So, $\omega_{\vec{v}}$ is a linear map from V^* to F □

Then we can define the canonical map $\pi : V \to V^{**}$ by $\pi(\vec{v}) = \omega_{\vec{c}}$

Theorem 3. *Given finite dimensional vector space V over field F the canonical map $\pi : V \to V^{**}$ is an isomorphism.*

Proof. Let $a, b \in F$, $\vec{u}, \vec{v} \in V$, and $S \in V^*$.
Consider.

$$\begin{aligned}\pi(a\vec{u}) + \pi(b\vec{v})) &= \omega_{a\vec{u}} + \omega_{b\vec{v}} \\ &= S(a\vec{u}) + S(b\vec{v}) && \forall S \in V^* \\ &= S(a\vec{u} + b\vec{b}) \\ &= \omega_{a\vec{u}+b\vec{v}} \\ &= \pi(a\vec{u} + b\vec{v})\end{aligned}$$

So, π is linear (operation preserving), so π is a homomorphism.
Finally consider $ker(\pi) = \{\vec{v} \in V | \pi(\vec{v}) = 0\}$ $\pi(\vec{v}) = 0$ implies $\omega_{\vec{v}} = 0$. That is, for all $\phi \in V^*$ such that $\phi(\vec{v}) = 0$. By $Proposition 2$, $ker(\pi) = \{0\}$. Then, $ker(\phi) = \{0\}$ implies that π is injective
By Proposition $1, dim(V^{**} = dimV^* = dimV$ and π is injective, so π is surjective. □

While we expect $V^{**} \simeq V$, it is notable that the canonical map is an isomorphism. That makes V^{**} and V reflexive algebras. [2]

3 Multi-Linear Forms

We can further extend the ideas embodied in the Dual Space to include multilinearity. So, as with Dual Spaces, we begin with (multi-) linear forms.

Given vector spaces $V_1, V_2, \ldots, V_n$, and W over field F we can define a map $\phi : V_1 \times \cdots \times V_n \to W$. If ϕ is linear on each term. ϕ is $n-$linear.

For a bi-linear map, $T : V_1 \times V_2 \to W$ is linear if, for $a, b \in F, \vec{u}, \vec{v} \in V_1$, and $\vec{x}, \vec{y} \in V_2, T(a(\vec{u} + b\vec{v}, \vec{x}) = aT(\vec{u}, \vec{x}) + bT(\vec{v}, \vec{x})$ and $T(\vec{u}, a\vec{x} + b\vec{y}) = aT(\vec{u}, \vec{x}) + bT(\vec{u}, \vec{y})$.

In the same way that V^* is a vector space if V is a vector space, the set of all $n-$linear maps from $V_1 \times \cdots \times V_n$ to W, which is denoted $\mathcal{L}(U, V; W)$. If $V_1, \ldots, V_n$ are vector spaces over field F and $\phi : V_1 \times \cdots \times V_n \to F$, ϕ is a $n-$linear form.

It is important to note that $V_1 \times \cdots \times V_n$ is a Cartesian product (not a direct product). The notation $\mathcal{L}^n(V_1 \times \cdots \times V_n; W)$ denotes the set of linear maps from the direct product $V_1 \times \cdots \times V_n$ to W. [2]

Lemma 4. *Given finite vector spaces V and W for all $T \in \mathcal{L}(V, W)$:*

- *There exists a $\varphi \in \mathcal{L}(V, W)$ such that $\varphi(\vec{v}) = \vec{0}$ for all $\vec{v} \in V$.*
- *$\vec{v} = \vec{0}$ if and only if, for all $\phi_i \in \mathcal{L}(V, W), \phi_i(\vec{v}) = \vec{0}$*

Proof. Let $O \in \mathcal{L}(V, W)$ be the zero transformation. Then, for all $\vec{v} \in V$, $O(\vec{v}) = \vec{0}$,

Then consider $\vec{0} \in V$ and $\phi \in \mathcal{L}(V, W)$.

$\phi(\vec{0}) = 0\phi(\vec{v}) = \vec{0}$

Finally, $O \in \mathcal{L}(V, W)$. $O(\vec{v}) = \vec{0}$ implies $\vec{v} = \vec{0}$. □

Theorem 5. *Given vector spaces V, U, and W over field F, $\mathcal{L}^2(V, U; W)$ is a vector space.*

Proof. First consider $Hom(U, Hom(V, W))$. By Theorem 1, $Hom(V, W)$ is a vector space. Similarly $Hom(U, Hom(V, W))$ is a vector space.

As an aside, $T : Hom(U, Hom(V, W))$ is $\mathcal{L}(U, \mathcal{L}(V, W)$.

Let $a, b \in F, \vec{u_i} \in U, \vec{v_i} \in V$, and $\phi_i \in \mathcal{L}(U, \mathcal{L}(V, W))$

Define map $T : \mathcal{L}(U, \mathcal{L}(V, W) \to \mathcal{L}^2(U, V; W)$ by $T(\phi) = \hat{\phi}(\vec{u}, \vec{v}) = (\phi(\vec{u}))(\vec{v})$, for all $\vec{u} \in U$ and $\vec{v} \in V$.

First, we check the bilinearity of $\hat{\phi}$

Consider,

$$\begin{aligned} \hat{\phi}(a\vec{u_1} + b\vec{u_2}, \vec{v}) &= (\phi(a\vec{u_1} + b\vec{u_2}))(\vec{v}) \\ &= (a\phi(\vec{u_1}))(\vec{v}) + (b\phi(\vec{u_2}))(\vec{v}) \qquad \text{By the linearity of } \phi \end{aligned}$$

And,

$$\begin{aligned} \hat{\phi}(\vec{u}, a\vec{v_1} + b\vec{v_2}) &= (\phi(\vec{u}))(a\vec{v_1} + b\vec{v_2}) \\ &= a(\phi(\vec{u}))(\vec{v}) + b(\phi(\vec{u}))(\vec{v_1}) \qquad \text{By the linearity of } \phi(\vec{u}) \end{aligned}$$

So, $\hat{\phi}$ is bilinear.

Consider,

$$\begin{aligned} aT(\phi_1) + bT(\phi_2) &= a(\phi_1(\vec{u}))(\vec{v}) + b(\phi_2(\vec{u}))(\vec{v}) && \text{for all } \vec{u} \in U, \vec{v} \in V \\ &= (a\phi_1(\vec{u}) + b\phi_2(\vec{u}))(\vec{v}) && \text{By the linearity of } \phi_i \\ &= T(a\phi_1 + b\phi_2) \end{aligned}$$

So, T is linear.

Consider, $\ker(T) = \{\phi \in \mathcal{L}(U, \mathcal{L}(V, W) : T(\phi) = \vec{0} = O\}$.

So, $O(\vec{u}, \vec{v}) = \hat{\phi}(\vec{u}, \vec{v}) = (\phi(\vec{u}))(\vec{v})$ (for all $\vec{u} \in U, \vec{v} \in V$.)

Therefore, $\phi(\vec{u}) = \vec{0}$ for all $\vec{u} \in U$.

Then, by Lemma 4, $\phi = \vec{0}$. So, $\ker(T) = \{\vec{0}\}$.

Then, T is injective.

Note $\dim(\ker(T)) = \dim(\{\vec{0}) = 0$

Consider, $Im(T) = \{T(\phi) \in \mathcal{L}(U, V; W) : \phi \in \mathcal{L}(U, \mathcal{L}(V, W))\}$.

Using $\dim(\mathcal{L}^2(U, V; W)) = \dim(Im(T)) - \dim(ker(T))$, we conclude that $\dim(Im(T)) = \dim(\mathcal{L}(U, V; W))$.

Therefore, T is surjective.

Therefore, T is an isomorphism and $\mathcal{L}(U, \mathcal{L}(V, W)) \simeq \mathcal{L}(U, V; W)$.

Therefore, $\mathcal{L}^2(U, V; W)$ is a vector space.

□

Corollary 3. *Given vector spaces U, V, and W over field F,*
$\dim(\mathcal{L}^2(U, V; W)) = \dim(U)\dim(V)\dim(W)$.

Proof. By Corollary 2, $\dim(\mathcal{L}(V, W)) = \dim(V)\dim(W)$.

Then, because $\mathcal{L}(V, W)$ is a vector space in its own right, we can apply $Corollary 2$ again.

So, $\dim(\mathcal{L}(U, \mathcal{L}(V, W))) =$
$\dim(U)(\dim(V)\dim(W)) = \dim(U)\dim(V)\dim(W)$

The, because $\mathcal{L}(U, \mathcal{L}(V, W)) \simeq \mathcal{L}^2(U, V; W)$,
$\dim(\mathcal{L}(U, V; W)) = \dim(\mathcal{L}(U, \mathcal{L}(V, W))) = \dim(U)\dim(V)\dim(W)$. □

We can similarly prove that $\mathcal{L}^n(V_1, \ldots, V_n; W)$ is a vector space.

Theorem 6. *Given finite vector spaces $V_1, \ldots, V_n$, and W over field F,*
$\mathcal{L}^n(V_1, \ldots, V_n; W)$ is a vector space.

Proof. We will prove Theorem 5 by induction on n for $n \geq 2$.

Let $\vec{v_i} \in V_i$ for $1 \leq i \leq n$

For $n = 2$, $T : \mathcal{L}(V_1, \mathcal{L}(V_2, W)) \to \mathcal{L}(V_1, V_2; W)$ defined by $T(\phi) = \hat{\phi}$ for $\hat{\phi}(\vec{v_1}, \vec{v_2}) = (\phi(\vec{v_1}))(\vec{v_2})$ is an isomorphism.

So, $\mathcal{L}(V_1, \mathcal{L}(V_2, W)) \simeq \mathcal{L}(V_1, V_2; W)$

Assume the induction hypothesis.

That is, for $n = k$:
$\mathcal{L}(V_1, \mathcal{L}^{k-1}(V_2, \ldots, V_k; W)) \simeq \mathcal{L}(V_1, \ldots, V_k; W)$

Therefore,
$\mathcal{L}(V_1, \mathcal{L}^{k-1}(V_2, \ldots, V_k; W)) \simeq \cdots \simeq \mathcal{L}(V_1, \mathcal{L}(V_2, \ldots, \mathcal{L}(V_k, W)) \ldots))$

Additionally,
$T_k : \mathcal{L}(V_1, \mathcal{L}^{k-1}(V_2, \ldots, V_k; W)) \to \mathcal{L}(V_1, \ldots, V_k; W)$ defined by
$T_k(\phi(\vec{v_1}, \vec{v_2}, \ldots \vec{v_{k-1}})) =$
$\hat{\phi}(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}) = (\phi(\vec{v_1}, \vec{v_2}, \ldots \vec{v_{k-1}}))(\vec{v_k})$ is an isomorphism.

Note that T_k is linear. Therefore,
$\mathcal{L}(V_1, \mathcal{L}^{k-1}(V_2, \ldots, V_k; W)) \simeq \mathcal{L}(V_1, \ldots, V_k; W)$

Consider the $n = k + 1$ case: $\mathcal{L}^{k+1}(V_1, \ldots, V_{k+1}; W)$.

Define map $T_{k+1} : \mathcal{L}(V_1, \mathcal{L}^k(V_2, \ldots, v_{k+1}; W)) \to \mathcal{L}^{k+1}(V_1, \ldots, V_k; W)$ by $T_{k+1}(\phi(\vec{v_1}, \vec{v_2}, \ldots \vec{v_k})) = \hat{\phi}(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_{k+1}}) = (\phi(\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}))(\vec{v_{k+1}})$.

The proof that T_{k+1} is almost identical as the proof in Theorem 5.

By the induction hypothesis $\phi(\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}) = (\varphi(\vec{v_1}, \vec{v_2}, \ldots \vec{v_{k-1}}))(\vec{v_k})$ for $\varphi \in \mathcal{L}^{k-1}(V_1, \ldots, V_{k-1})$.

That is, $T_k(\varphi) = \phi$. So, $T_{k+1}(\phi) = (T_k(\varphi))(\vec{v_{k+1}})$

Consider $ker(T_{k_1}) = \{\phi \in \mathcal{L}(V_1, \mathcal{L}^k(V_2, \ldots, v_k; W)) : T_{k+1}(\phi) = \vec{0}\}$.

For all $\vec{v_1} \in V_1, \ldots \vec{v_{k+1}} \in V_{k+1}$ and for all $\phi \in \mathcal{L}(V_1, \mathcal{L}^k(V_2, \ldots, v_k; W))$,

$$\begin{aligned}\vec{0} &= \hat{\phi}(\vec{v_1}, \ldots, \vec{v_{k+1}}) \\ &= (\phi(\vec{v_1}, \ldots, \vec{v_k}))(\vec{v_{k+1}}) \\ &= (T_k(\varphi(\vec{v_1}, \ldots, \vec{v_k}))(\vec{v_{k+1}})\end{aligned}$$

So, $T_k(\varphi(\vec{v_1}, \ldots, \vec{v_k}) = \vec{0}$

By the induction hypothesis, T_k is an isomorphism. So, $ker(T_k) = \{\vec{0}\}$.

Therefore, $ker(T_{k+1}) = \{\vec{0}\}$. So, T_{k+1} is injective.

Then, because $dim(\mathcal{L}(V_1, \mathcal{L}^k(V_2, \ldots, V_{k+1}; W)))$
$= dim(Im(T_{k+1})) - dim(ker(T_{k+1}))$, we conclude that
$dim(Im(T_{k+1})) = dim(\mathcal{L}(U, V; W))$.

Therefore, T_{k+1} is surjective.

Therefore, T_{k+1} is an isomorphism and $\mathcal{L}(V_1, \mathcal{L}^k(V_2, \ldots, v_{k+1}; W))$
$\simeq \mathcal{L}^{k+1}(V_1, \ldots, V_k; W)$.

□

4 Tensor Products

Now, we can begin to understand tensors and tensor products. We start with the simplest case: two finite vector spaces.

Given finite vector spaces U, V, and W over a field F, we can construct another vector space $U \otimes V$ with bilinear map $\pi : U \times V \to U \otimes V$. If for any W and every $T \in \mathcal{L}^2(U, V; W)$, there exists a unique linear map, $\hat{T} : U \otimes V \to W$ such that $T = \pi \circ \hat{T}$, $U \otimes V$ is a tensor product. [2]

First we must prove that tensor products exist. As we will later see, the tensor product of two vector spaces is unique up to isomorphism. For this reason, our choice of "W" is unimportant. [4]

Theorem 7. *Given finite vector spaces U, V, and W, the tensor product of U and V exists.*

Proof. The proof of this theorem goes well beyond the scope of this paper. [5] □

Tensor products are unique up to isomorphism.

Theorem 8. *Given two finite vector spaces U and V, $U \otimes V$ is unique up to isomorphism.*

Proof. Let $(U \otimes V)_1, \phi_1$ and $(U \otimes V)_2, \phi_2$ be two tensor products of U, V.

First note that $(U \otimes V)_1$ and $(U \otimes V)_2$ are Vector Spaces. So, we can define a tensor product "to" them.

By definition, we can obtain a unique linear map $T : (U \otimes V)_1 \to (U \otimes V)_2$ such that $\phi_2 = T \circ \phi_1$.

Similarly, we can obtain a unique linear map $S : (U \otimes V)_2 \to (U \otimes V)_1$ such that $\phi_1 = S \circ \phi_2$.

Therefore, $(S \circ T) \circ \phi_1 = S \circ \phi_2 = \phi_1$.

Then, by the uniqueness of T and S, $S \circ T = id$. Therefore, $S = T^{-1}$ and T is an isomorphism. So, $(U \otimes V)_1, \phi_1 \simeq (U \otimes V)_2, \phi_2$. $\square$

Because a Tensor Product of two vectors spaces is unique up to isomorphism, we refer to *the* Tensor Product.

It should then be obvious that $\mathcal{L}(U \otimes V; W) \simeq \mathcal{L}^2(U, V; W)$

This allows us to "set" W to the field of scalars F.

It then becomes more convenient to use V, V^*, and V^{**}.

For the next theorem, it is necessary to understand $\vec{v} \otimes \vec{w}$. For the purposes of this paper, it is adequate to say that $\vec{v} \otimes \vec{w}$ is the outer product of $\vec{v}$ and $\vec{w}$.

As a concrete example, consider $\begin{bmatrix} [1 \\ 2 \\ 3] \end{bmatrix} \in \mathbb{R}^3$ and $\begin{bmatrix} [1 \\ 2] \end{bmatrix} \in \mathbb{R}^2$.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

Theorem 9. *Given $\{\vec{u_1}, \ldots, \vec{u_n}\}$ a basis of finite vector space U and $\{\vec{v_1}, \ldots, \vec{v_m}\}$ a basis of finite vector space V, the set $\{\vec{u_i} \otimes \vec{v_j}\}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ forms a basis of $U \otimes V$.*

Proof. Again, the proof of this theorem goes well beyond the scope of this paper. [5] $\square$

Corollary 4. *Given finite dimensional Vector Spaces U and V,* $\dim(U \otimes V) = \dim(U)\dim(V)$

Proof. If $\{\vec{u_i} \otimes \vec{v_j}\}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ forms a basis f $U \otimes V$, $\dim(U \otimes V) = |\{\vec{u_i} \otimes \vec{v_j}\}| = \dim(U)\dim(V)$. $\square$

We can then begin to explore the behavior of $\otimes$. [4]

Proposition 3. *Given finite vector spaces U and V where $\vec{u} \in U$ and $\vec{v} \in V$, if $\vec{u} \otimes \vec{v} = \vec{0}$, then, $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.*

Proposition 4. *Given finite dimensional vector spaces U and V, $U \otimes V \simeq V \otimes U$.*

Proposition 5. *Given finite vector spaces $U, V,$ and W then,* $U \otimes (V \otimes W) \simeq (U \otimes V) \otimes W$.

5 Conclusion

At its core, the tensor product generalizes the concept of multi-linearity to form a "new" finite vector space from a given set of finite vector spaces with the operation of multi-linear composition.

The elements of a tensor product are called tensors. In general, tensors are generalizations of vectors.

In the same way that many systems (physical or otherwise) require more than a simple scalar to be described, some systems require more than a vector to be described.

Tensors, as a generalization of vectors, allow for these systems to be described.

For example, the stress in a $3-$Dimensional object requires 2 "components" per direction. So, stress lends itself to being described by a tensor with 2 components per basis, making it a $(2, 0)-$Tensor.

As a resultant, Tensors are widely applicable across a variety of fields.

6 Sources

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