Math 290 Name: Key Dr. Beezer
Exam 6 Spring 2020
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features). P_n is the vector space of polynomials with degree n or less, and $M_{m,n}$ is the vector space of $m \times n$ matrices.

1. Use the *definition* of a linear transformation to verify that T is a linear transformation. (15 points)

$$T\colon \mathbb{C}^2 \to \mathbb{C}^2 \qquad T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 2a+b \\ a-b \end{bmatrix}$$

① $T(\underset{\sim}{x}+\underset{\sim}{y}) = T\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} a_1+a_2 \\ b_1+b_2 \end{bmatrix}\right) = \begin{bmatrix} 2(a_1+a_2)+(b_1+b_2) \\ (a_1+a_2)-(b_1+b_2) \end{bmatrix}$

$= \begin{bmatrix} (2a_1+b_1)+(2a_2+b_2) \\ (a_1-b_1)+(a_2-b_2) \end{bmatrix} = \begin{bmatrix} 2a_1+b_1 \\ a_1-b_1 \end{bmatrix} + \begin{bmatrix} 2a_2+b_2 \\ a_2-b_2 \end{bmatrix} = T(\underset{\sim}{x})+T(\underset{\sim}{y})$

② $\alpha \in \mathbb{C}$, $T(\alpha\underset{\sim}{x}) = T\left(\alpha\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}\right) = \begin{bmatrix} 2(\alpha a)+\alpha b \\ \alpha a-\alpha b \end{bmatrix}$

$= \begin{bmatrix} \alpha(2a+b) \\ \alpha(a-b) \end{bmatrix} = \alpha\begin{bmatrix} 2a+b \\ a-b \end{bmatrix} = \alpha T(\underset{\sim}{x})$

So, by Definition LT, T is a linear transformation

2. Find a basis for the range of T, $\mathcal{R}(T)$. (20 points)

$$T\colon \mathbb{C}^3 \to P_2 \qquad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a-2b-c)+(2a+b+3c)x+(3a+3c)x^2$$

Use a basis of $\mathbb{C}^3$ ($\{\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3\}$) and Theorem SSRLT to get a spanning set:

$T(\underset{\sim}{e}_1) = 1+2x+3x^2$, $T(\underset{\sim}{e}_2) = -2+x$, $T(\underset{\sim}{e}_3) = -1+3x+3x^2$

Then $R(T) = \langle\{1+2x+3x^2, -2+x, -1+3x+3x^2\}\rangle$

BUT this set is not linearly independent. Note: the third polynomial is the sum of the first two. So remove it.

Basis: $\{1+2x+3x^2, -2+x\}$

3. Consider the linear transformation S. (35 points)

$S\colon M_{1,3} \to P_2 \qquad S\left(\begin{bmatrix} a & b & c \end{bmatrix}\right) = (2a+7b+6c) + (a+4b+4c)x + (-b-c)x^2$

(a) Demonstrate that S is invertible.

$[a\ b\ c] \in K(S)$ if a,b,c is a solution to a homogeneous system:

$$\begin{bmatrix} 2 & 7 & 6 \\ 1 & 4 & 4 \\ 0 & -1 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} I_3$$

coefficient matrix

So $K(S) = \{\underset{\sim}{0}\}$; $n(S) = 0$ & S injective

$r(S) + n(S) = \dim(M_{13}) = 1 \cdot 3 = 3$

$\Rightarrow r(S) = 3$, which equals $\dim(P_2) = 3$

So S is surjective

Theorem ILTIS $\Rightarrow$ S is invertible.

(b) Find a formula for $S^{-1}\colon P_2 \to M_{1,3}$.

A basis for P_2 is $\{1, x, x^2\}$. Preimages?

$S^{-1}(1) \Rightarrow$ solve system $\left[\begin{array}{ccc|c} 2 & 7 & 6 & 1 \\ 1 & 4 & 4 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{c|c} I_3 & \begin{matrix} 0 \\ 1 \\ -1 \end{matrix} \end{array}\right] \Rightarrow S^{-1}(1) = \{[0\ 1\ -1]\}$

Similarly $S^{-1}(x) = \{[1\ -2\ 2]\}$ & $S^{-1}(x^2) = \{[4\ -2\ 1]\}$. Then

$S^{-1}(a+bx+cx^2) = aS^{-1}(1) + bS^{-1}(x) + cS^{-1}(x^2) = a[0\ 1\ -1] + b[1\ -2\ 2] + c[4\ -2\ 1]$

$= [0\ a\ -a] + [b\ -2b\ 2b] + [4c\ -2c\ c]$

$= [b+4c \quad a-2b-2c \quad -a+2b+c]$

(c) Compute the composition $S^{-1} \circ S$ and explain why this is a partial check on your work above.

$(S^{-1} \circ S)([a\ b\ c]) = S^{-1}(S([a\ b\ c])) = S^{-1}((2a+7b+6c) + (a+4b+4c)x + (-b-c)x^2)$

$= [a+4b+4c+4(-b-c) \quad 2a+7b+6c-2(a+4b+4c)-2(-b-c) \quad -(2a+7b+6c)+2(a+4b+4c)+(-b-c)]$

Careful!

$= [a \quad b \quad c]$ So $S^{-1} \circ S = I_{M_{13}}$, as expected

A ~~full~~ check would show $S \circ S^{-1} = I_{P_2}$

4. The linear transformation R is not injective (you may assume this). Find two different vectors, $\mathbf{x}$ and $\mathbf{y}$, such that $R(\mathbf{x}) = R(\mathbf{y})$. (15 points)

$$R\colon \mathbb{C}^4 \to P_3 \qquad R\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right) = (b + c - 2d) + (-a + 3c + 5d)x + (2b + 3c - 3d)x^2 + (4b + 8c - 4d)x^3$$

By Theorem KILT, the kernel will not be trivial. Find this first:

$0 + 0x + 0x^2 + 0x^3 = \underset{\sim}{0} = R\left(\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}\right)$ leads to a homogeneous system w/ coefficient matrix,

$$\begin{bmatrix} 0 & 1 & 1 & -2 \\ -1 & 0 & 3 & 5 \\ 0 & 2 & 3 & -3 \\ 0 & 4 & 8 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $a = 2$, $b = 3$, $c = -1$, $d = 1$ is a non-trivial solution.

$$\text{So } R\left(\begin{bmatrix} 2 \\ 3 \\ -1 \\ -1 \end{bmatrix}\right) = 0 + 0x + 0x^2 + 0x^3 \underset{\text{LTTZZ}}{=} R\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

$\underset{\sim}{x}$ $\underset{\sim}{y}$

5. Suppose that $T\colon U \to V$ and $S\colon U \to V$ are linear transformations which agree on a basis of U. That is, for some basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ of U, $T(\mathbf{u}_i) = S(\mathbf{u}_i)$ for $1 \le i \le n$. Prove that T and S are equal functions. (15 points)

$T = S$ means $T(\underset{\sim}{u}) = S(\underset{\sim}{u})$ for every $\underset{\sim}{u} \in U$

$$\begin{aligned} T(\underset{\sim}{u}) &= T(a_1 \underset{\sim}{u_1} + a_2 \underset{\sim}{u_2} + \cdots + a_n \underset{\sim}{u_n}) && \text{B spans } U \\ &= a_1 T(\underset{\sim}{u_1}) + a_2 T(\underset{\sim}{u_2}) + \cdots + a_n T(\underset{\sim}{u_n}) && \text{LTLC} \\ &= a_1 S(\underset{\sim}{u_1}) + a_2 S(\underset{\sim}{u_2}) + \cdots + a_n S(\underset{\sim}{u_n}) && \text{Hypothesis} \\ &= S(a_1 \underset{\sim}{u_1} + a_2 \underset{\sim}{u_2} + \cdots + a_n \underset{\sim}{u_n}) && \text{LTLC} \\ &= S(\underset{\sim}{u}) \end{aligned}$$