Math 290 Name: Key Dr. Beezer
Exam 4 Spring 2020
Chapter VS

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. The set $S$ below is from the vector space $P_3$, polynomials with degree at most 3. Determine, with explanation, if $S$ is linearly independent. (15 points)

$S = \{x^3 + 3x^2 + 2x - 1,\ x^3 + 2x^2 + x - 2,\ x^3 + 4x^2 + 3x\}$

One way or another, establish that

$$2(x^3+3x^2+2x-1) + (-1)(x^3+2x^2+x-2) + (-1)(x^3+4x^2+3x)$$
$$= 0x^3 + 0x^2 + 0x + 0 = \underset{\sim}{0}$$

So we have a nontrivial RLD, and $S$ is linearly dependent.

2. The set $T$ below is from the vector space $M_{22}$, comprised of $2 \times 2$ matrices. Determine, with explanation, if $T$ spans $M_{22}$. (15 points)

$T = \left\{ \begin{bmatrix} -5 & 3 \\ -3 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}, \begin{bmatrix} 4 & -4 \\ 3 & -5 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \right\}$ $\dim(M_{22}) = 2 \cdot 2 = 4$ by a theorem

$T$ has the "right" size to be a basis. We can establish linear independence & Theorem G will give spanning "for free".

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \underset{\sim}{0} = a_1 \begin{bmatrix} -5 & 3 \\ -3 & 3 \end{bmatrix} + a_2 \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix} + a_3 \begin{bmatrix} 4 & -4 \\ 3 & -5 \end{bmatrix} + a_4 \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -5a_1 + a_2 + 4a_3 & 3a_1 - 2a_2 - 4a_3 + a_4 \\ -3a_1 + a_2 + 3a_3 - a_4 & 3a_1 - 3a_2 - 5a_3 + 2a_4 \end{bmatrix}$$

$\Rightarrow$ homogeneous system w/ 4 equations & 4 variables $\Rightarrow$ coefficient matrix $\xrightarrow{\text{RREF}}$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ So the only solution is $a_1 = a_2 = a_3 = a_4 = 0$ & Thus $T$ is linearly independent

3. Determine the dimension of the subspace $W$ of the vector space $M_{22}$, comprised of $2 \times 2$ matrices. (20 points)

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a - 2c + d = 0,\ b + 4c - 3d = 0 \right\} \subseteq M_{22}$$

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a = 2c - d,\ b = -4c + 3d \right\} = \left\{ \begin{bmatrix} 2c-d & -4c+3d \\ c & d \end{bmatrix} \middle| \, c, d \in \mathbb{C} \right\}$$

$$= \left\{ c \begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix} \middle| \, c, d \in \mathbb{C} \right\} = \left\langle \left\{ \begin{bmatrix} 2 & -4 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 0 & 1 \end{bmatrix} \right\} \right\rangle$$

So we have a spanning set. Check the linear independence of this set (easy w/ entries in bottom rows). So a <u>basis</u> w/ two elements $\Rightarrow \dim(W) = 2$.

4. Demonstrate the use of the three parts of Theorem TSS to prove that $U$ is a subspace of $\mathbb{C}^2$. (20 points)

$$U = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| \, 2a - b = 0 \right\} \subseteq \mathbb{C}^2$$

1) $\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in U$ because $2(0) - 0 = 0$.

2) Suppose $\underset{\sim}{x} = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \in U$ & $\underset{\sim}{y} = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \in U$. Then $2a_1 - b_1 = 0$ & $2a_2 - b_2 = 0$.

$\underset{\sim}{x} + \underset{\sim}{y} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$. Check

$2(a_1 + a_2) - (b_1 - b_2) = (2a_1 - b_1) + (2a_2 - b_2) = 0 + 0 = 0 \Rightarrow \underset{\sim}{x} + \underset{\sim}{y} \in U$

3) Suppose $\underset{\sim}{x} = \begin{bmatrix} a \\ b \end{bmatrix} \in U$. Then $2a - b = 0$. Let $\alpha \in \mathbb{C}$

Then $\alpha \underset{\sim}{x} = \begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}$. Check

$2(\alpha a) - (\alpha b) = \alpha(2a - b) = \alpha \cdot 0 = 0$, so $\alpha \underset{\sim}{x} \in U$.

5. Suppose that $S$ is a finite set of vectors from the vector space $V$. Prove the third part of Theorem TSS showing that the span of $S$ has scalar multiplication closure. In other words, show that if $\alpha \in \mathbb{C}$ and $\mathbf{x} \in \langle S \rangle$ then $\alpha\mathbf{x} \in \langle S \rangle$, without employing the theorem that says the span is a subspace. (15 points)

Write $S = \{\underset{\sim}{v}_1, \underset{\sim}{v}_2, \dots, v_m\}$. Then there are scalars,

$\underset{\sim}{x} \in \langle S \rangle \Rightarrow \quad \underset{\sim}{x} = a_1\underset{\sim}{v}_1 + a_2\underset{\sim}{v}_2 + \cdots + a_m v_m$. Now

$$\alpha\underset{\sim}{x} = \alpha(a_1\underset{\sim}{v}_1 + a_2\underset{\sim}{v}_2 + \dots + a_m\underset{\sim}{v}_m) = (\alpha a_1)\underset{\sim}{v}_1 + (\alpha a_2)\underset{\sim}{v}_2 + \dots + (\alpha a_m)v_m$$

scalars

this qualifies $\alpha\underset{\sim}{x}$ for membership in $\langle S \rangle$.

6. Suppose $B = \{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis of the vector space $V$. Prove that $C = \{\mathbf{v_1} + \mathbf{v_2}, 2\mathbf{v_1} + \mathbf{v_2}\}$ is also a basis of $V$. (15 points)

First, check that $C$ is linearly independent.

$$\underset{\sim}{0} = a_1(\underset{\sim}{v}_1 + v_2) + a_2(2\underset{\sim}{v}_1 + \underset{\sim}{v}_2) = (a_1 + 2a_2)\underset{\sim}{v}_1 + (a_1 + a_2)\underset{\sim}{v}_2$$

$B$ linearly independent $\Rightarrow \left.\begin{array}{l} a_1 + 2a_2 = 0 \\ a_1 + a_2 = 0 \end{array}\right\}$ a homogeneous system w/ only the solution $a_1 = a_2 = 0$.

So $C$ is linearly independent.

$\dim(B) = 2$ (given a basis of size 2) and $|C| = 2$ so Theorem G says $C$ spans $V$. Thus $C$ is a basis.

OR establish spanning directly. Grab $\underset{\sim}{x} \in V$

$$\underset{\sim}{x} = \underbrace{b_1\underset{\sim}{v}_1 + b_2\underset{\sim}{v}_2}_{B \text{ spans } V} = b_1\left(-(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) + (2\underset{\sim}{v}_1 + \underset{\sim}{v}_2)\right) + b_2\left(2(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) + (2\underset{\sim}{v}_1 + \underset{\sim}{v}_2)\right)$$

$$= (-b_1 + 2b_2)(\underset{\sim}{v}_1 + \underset{\sim}{v}_2) + (b_1 + b_2)(2\underset{\sim}{v}_1 + \underset{\sim}{v}_2)$$

a linear combination of vectors from $C$