Math 290
Exam 2
Chapter V

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Determine if the vector $\mathbf{x}$ is in the span of S, $\mathbf{x} \in \langle S \rangle$, with a complete and careful explanation. (15 points)

$$\mathbf{x} = \begin{bmatrix} -3 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 2 \\ -3 \end{bmatrix} \right\}$$

① Are there scalars a_1, a_2, a_3 with $a_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 7 \\ -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \\ 1 \end{bmatrix}$?
(Definition SSCV)

② Solutions to the system w/ augmented matrix? (Theorem SLSLC)

$$\left[\begin{array}{ccc|c} 1 & -5 & 7 & -3 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 2 & 2 \\ -1 & 1 & -3 & 1 \end{array}\right] \xrightarrow[\text{RREF}]{③} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

④ RCLS ⇒ system inconsistent
⇒ no solution
⇒ $\mathbf{x} \notin \langle S \rangle$

2. Determine if the set T is linearly independent, with a complete and careful explanation. (15 points)

$$T = \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ -8 \\ 6 \\ -6 \end{bmatrix} \right\}$$

① To use Theorem LIVRN make these vectors the columns of a matrix.

$$A = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 5 & -8 \\ -1 & -4 & 6 \\ 2 & 5 & -6 \end{bmatrix} \xrightarrow{② \text{ RREF}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

③ $2 = r < n = 3$, so by Theorem LIVRN, T is linearly dependent.

3. Find a linearly independent set R that spans the null space of A, that is, $\mathcal{N}(A) = \langle R \rangle$. (20 points)

$$A = \begin{bmatrix} 0 & 1 & 0 & 2 & -1 \\ -1 & 2 & -2 & 7 & 5 \\ -1 & 2 & -1 & 6 & 1 \\ 0 & 2 & -2 & 6 & 6 \end{bmatrix}$$

We can apply Theorem BNS to get the requested set

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$r=3$; two free variables, x_4 & x_5 if we discuss $LS(A, \underline{0})$

$$N(A) = \left\langle \left\{ \begin{bmatrix} \\ \\ \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \underbrace{\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \\ 0 \\ 1 \end{bmatrix} \right\}}_{R\text{, linearly independent by BNS}} \right\rangle$$

"pattern of zeros & ones"

4. Find a linearly independent set K whose span is equal to the span of L, $\langle K \rangle = \langle L \rangle$. (20 points)

$$L = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 3 \\ -6 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 4 \end{bmatrix} \right\}$$

We will apply Theorem BS to get the requested set.

Make a matrix whose columns are the vectors of L.

$$C = \begin{bmatrix} 1 & 0 & 6 & -2 & -3 & -5 \\ 0 & 1 & 3 & 1 & 3 & -1 \\ 0 & -2 & -6 & -1 & -6 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 6 & 0 & -3 & -1 \\ 0 & 1 & 3 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 & 0 & 2 \end{bmatrix}$$

pivot columns are 1, 2, 4. So K is vectors 1,2,4 of L:

$$K = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

5. Provide a proof for Property DSAC: if α, $\beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$. (15 points)

For $1 \leq i \leq m$

$$\begin{aligned}[(\alpha+\beta)\underset{\sim}{u}]_i &= (\alpha+\beta)[\underset{\sim}{u}]_i && \text{Defn CVSM}\\ &= \alpha[\underset{\sim}{u}]_i + \beta[\underset{\sim}{u}]_i && \text{Property DCN}\\ &= [\alpha\underset{\sim}{u}]_i + [\beta\underset{\sim}{u}]_i && \text{Defn CVSM}\\ &= [\alpha\underset{\sim}{u} + \beta\underset{\sim}{u}]_i && \text{Defn CVA}\end{aligned}$$

So, by Definition CVE, we have $(\alpha+\beta)\underset{\sim}{u} = \alpha\underset{\sim}{u} + \beta\underset{\sim}{u}$.

6. Suppose that S is a set of vectors, $S \subseteq \mathbb{C}^m$. Prove that if $\mathbf{x}$, $\mathbf{y} \in \langle S \rangle$, then $\mathbf{x} + \mathbf{y} \in \langle S \rangle$. (15 points)

Write $S = \{\underset{\sim}{v_1}, \underset{\sim}{v_2}, \dots \underset{\sim}{v_n}\}$

Then $\underset{\sim}{x} = a_1\underset{\sim}{v_1} + a_2\underset{\sim}{v_2} + \cdots + a_n\underset{\sim}{v_n}$ for some scalars $a_1, \dots, a_n$

and $\underset{\sim}{y} = b_1\underset{\sim}{v_1} + b_2\underset{\sim}{v_2} + \cdots + b_n\underset{\sim}{v_n}$ for some scalars $b_1, \dots, b_n$.

So

$$\begin{aligned}\underset{\sim}{x}+\underset{\sim}{y} &= (a_1\underset{\sim}{v_1} + \cdots + a_n\underset{\sim}{v_n}) + (b_1\underset{\sim}{v_1} + \cdots + b_n\underset{\sim}{v_n})\\ &= a_1\underset{\sim}{v_1} + b_1\underset{\sim}{v_1} + a_2\underset{\sim}{v_2} + b_2\underset{\sim}{v_2} + \cdots + a_n\underset{\sim}{v_n} + b_n\underset{\sim}{v_n}\\ &= (a_1+b_1)\underset{\sim}{v_1} + (a_2+b_2)\underset{\sim}{v_2} + \cdots + (a_n+b_n)\underset{\sim}{v_n}\end{aligned}$$

So $\underset{\sim}{x}+\underset{\sim}{y}$ is a linear combination of S, thus $\underset{\sim}{x}+\underset{\sim}{y} \in \langle S \rangle$