Math 290
Exam 1
Chapter SLE

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices, except in the question that asks you to row-reduce without Sage. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Solve the following two systems of linear equations and express the solutions for each as a set of column vectors. (25 points)

(a) Form augmented matrix, row-reduce & analyze

$$\begin{aligned} 3x_1 - 2x_2 - 3x_3 &= 2 \\ 4x_1 - 3x_2 - 7x_3 &= 3 \\ 2x_1 - x_2 \qquad &= 2 \\ x_1 - 2x_2 - 8x_3 &= -3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & -2 & -3 & 2 \\ 4 & -3 & -7 & 3 \\ 2 & -1 & 0 & 2 \\ 1 & -2 & -8 & -3 \end{array}\right] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Column 4 is not a pivot column, RCLS ⇒ system is consistent
$n=3$, $r=3$ ⇒ $n-r=0$ free variables ⇒ unique solution
$x_1=5$, $x_2=8$, $x_3=-1$; as a set: $\left\{\begin{bmatrix} 5 \\ 8 \\ -1 \end{bmatrix}\right\}$

(b) Form augmented matrix, row-reduce & analyze

$$\begin{aligned} -2x_1 + 5x_2 + 8x_3 - 3x_4 &= -2 \\ x_1 - 3x_2 - 5x_3 + 2x_4 &= 1 \\ 3x_2 + 6x_3 - 3x_4 &= 1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} -2 & 5 & 8 & -3 & -2 \\ 1 & -3 & -5 & 2 & 1 \\ 0 & 3 & 6 & -3 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Column 5 is a pivot column, so RCLS ⇒ inconsistent

Solution as a set: $\{\} = \emptyset$

2. Is the matrix B singular or not? Provide a justification for your answer. (10 points)

$$B = \begin{bmatrix} -1 & 0 & 3 & -1 \\ 1 & -1 & 1 & -5 \\ 1 & 0 & -4 & 2 \\ 1 & -2 & 1 & -7 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq I_4$$

So by Theorem NMRII, B is singular.

3. Without using Sage, find a matrix B in reduced row-echelon form which is row-equivalent to A. It is especially important to show all of your work, so it is clear you have not used Sage. (15 points)

$$A = \begin{bmatrix} 1 & 2 & 8 & 3 \\ 1 & 1 & 5 & 1 \\ 2 & 3 & 13 & 4 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & 8 & 3 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -2 \end{bmatrix} \xrightarrow{-R_2} \begin{bmatrix} 1 & 2 & 8 & 3 \\ 0 & 1 & 3 & 2 \\ 0 & -1 & -3 & -2 \end{bmatrix}$$

$$\xrightarrow[-2R_2+R_1]{1R_2+R_3} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. Each part below gives some specific information about a system of linear equations. Say as much as you can about the solution set of the system, with explanations (such as indicating definitions and theorems which apply). (20 points)

(a) The system is consistent and has more equations than variables.

consistent $\Rightarrow$ system has one or infinitely many solutions (definition)

(b) The system is homogenous and has more variables than equations.

Theorem HMVEI $\Rightarrow$ infinitely many solutions

homogeneous $\Rightarrow$ $\underset{\sim}{0}$ is one of the solutions

(c) The system is homogenous and has the same number of variables as equations.

homogeneous $\Rightarrow$ $\underset{\sim}{0}$ is $\underline{\underline{\underline{a}}}$ solution.

There could be <u>more</u> but we cannot say.

(d) The coefficient matrix is nonsingular.

Theorem NMUS $\Rightarrow$ a unique solution.

(e) The system is homogenous and the coefficient matrix is nonsingular.

$$\left.\begin{array}{l}\text{Theorem NMUS} \Rightarrow \text{a unique solution} \\ \text{homogenous} \Rightarrow \underset{\sim}{0} \text{ is a solution}\end{array}\right\} \Rightarrow \begin{array}{l}\text{Solution set} \\ \text{is } \{\underset{\sim}{0}\}.\end{array}$$

5. Compute the null space of the matrix B, $\mathcal{N}(B)$. (15 points)

$$B = \begin{bmatrix} 1 & 3 & 3 & -1 \\ -3 & -8 & -8 & 2 \\ -1 & -7 & -7 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C$$

$N(B)$ = solutions to $LS(B, \underset{\sim}{0})$ = Solutions to $LS(C, \underset{\sim}{0})$

$$\begin{aligned} x_1 &= -2x_4 \\ x_2 &= -x_3 + x_4 \end{aligned} \Rightarrow N(B) = \left\{ \begin{bmatrix} -2x_4 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} \,\middle|\, x_3, x_4 \in \mathbb{C} \right\}$$

6. Consider the following theorem (which you are to *use*, but not *prove*): Suppose that

$$x_1 = a_1, x_2 = a_2, x_3 = a_3, \ldots, x_n = a_n \qquad \text{and} \qquad x_1 = b_1, x_2 = b_2, x_3 = b_3, \ldots, x_n = b_n$$

are two solutions to $\mathcal{LS}(A, \mathbf{b})$. Then a solution to the homogenous system $\mathcal{LS}(A, \mathbf{0})$ is

$$x_1 = a_1 - b_1, x_2 = a_2 - b_2, x_3 = a_3 - b_3, \ldots, x_n = a_n - b_n.$$

Use this theorem and the definition of a nonsingular matrix to prove: If A is a nonsingular matrix, then any solution to $\mathcal{LS}(A, \mathbf{b})$ is unique. (15 points)

Assume A is nonsingular & then suppose $LS(A, \underset{\sim}{b})$ has two solutions. Then

① $LS(A, \underset{\sim}{0})$ has only the trivial solution, $\underset{\sim}{x} = \underset{\sim}{0}$

② $LS(A, \underset{\sim}{0})$ has the solution

These two solutions must be the same. So

$0 = a_1 - b_1, \quad 0 = a_2 - b_2, \quad 0 = a_3 - b_3, \ldots, 0 = a_n - b_n$

$\Rightarrow a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3, \ldots, a_n = b_n$

So the "two" solutions are really the same.