# Cayley Graphs

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**Abstract**

In this paper, we study Cayley graphs and their properties as representations of groups. We examine both the construction of the Cayley graph of a group given an inverse-closed subset not containing the identity, and the identification of arbitrary graphs as Cayley graphs. We finish by showing that the components of a Cayley graph represent cosets of a subgroup.

# 0 Colophon



# 1 Introduction

When studying groups, we are often particularly interested in the set of subgroups of a particular group. We might also be interested in graphical representations of groups. One particularly informative graphical representation, based on group actions, is the Cayley graph of a group $G$ relative to some inverse-closed subset of $G \setminus e$, where $e$ is the identity of $G$. For small finite groups, in particular, Cayley graphs are an efficient way to see how $G$ acts on itself [5]. In addition, Cayley graphs can be used to analyze the different words of a generating subset of a group, i.e. products of powers of elements of a generating set [6].

## 1.1 Preliminaries

We assume that the reader is familiar with undergraduate abstract algebra, as well as basic graph theory terminology.

When we refer to a graph in this paper, we refer to a *simple* graph, i.e. one with no loops, only undirected edges, only unweighted edges, and no more than one edge directly connecting any two vertices.

If $S$ is a generating set of a finite group $G$, then we write $G = \langle S \rangle$. Any element $g \in G$ may be written as a product of some finite list of (not necessarily distinct) elements from $S$, e.g. $g = s_1 s_2 \cdots s_k$ We call $s_1, s_2, \ldots, s_k$ a **word** [6].

# 2 Cayley Graphs and Group Actions

Recall Cayley's Theorem, which says that any group can be represented as a permutation group, where the objects being permuted are the group elements, and the action performed by each element is the group operation, either from the left or the right. Considering multiplication on the left, we can define the **left regular representation** of $G$, $\lambda_g : G \mapsto G$, by $\lambda_g(x) = gx$ [1].

## 2.1 Introducing Cayley Graphs

For the rest of this paper, unless otherwise specified, $G$ is an arbitrary group and $C$ is an inverse-closed subset of $G \setminus e$.

**Definition 2.1** A graph $\Gamma$ is a **Cayley graph** if its vertex set is a group $G$ and its edge set is

$$E(\Gamma) = \{\{g, h\} \mid g, h \in G, hg^{-1} \in C\},$$

where $C$ is an inverse-closed subset of $G \setminus e$. We write $\Gamma = \Gamma(G, C)$, and we say that $\Gamma(G, C)$ is the **Cayley graph of $G$ relative to $C$**[2]. $\diamondsuit$

**Remark 2.2** Some definitions require that $C$ be a generating set for $G$ [3]. Others lack the requirement that $C$ is inverse-closed, and thus define a *directed*, rather than undirected Cayley graph. Cayley himself, in defining these graphs, specified that each edge $\{g, h\}$ be colored based on which $c \in C$ satisfies $h = cg$ [4]. Our working definition defines an undirected, uncolored graph [2].

Note that many groups have more than one Cayley graph. Here are two examples of Cayley graphs for the cyclic group of order 8, $\mathbb{Z}_8$. (We will revisit these later.)

**Example 2.3 Subset $C$ generates $\mathbb{Z}_8$.** Since $\mathbb{Z}_8 = \langle 1 \rangle$, let $C = \{1, -1\} = \{1, 7\}$. Then $\Gamma(\mathbb{Z}_8, C)$ is the cycle graph on 8 vertices, $C_8$. That is, $V(\Gamma) = \mathbb{Z}_8$, and $E(\Gamma) = \{\{0, 1\}, \{1, 2\}, \ldots, \{6, 7\}, \{0, 7\}\}$. Notice that this graph has just one component. $\square$

**Example 2.4 Subset $C^*$ does not generate $\mathbb{Z}_8$.** Let $C^* = \{2, 6\}$. This time, notice that $C^*$ generates the subgroup of $\mathbb{Z}_8$ isomorphic to $\mathbb{Z}_4$, rather than the entire group $\mathbb{Z}_8$. Then $\Gamma(\mathbb{Z}_8, C^*)$ has edge set $E(\Gamma) = \{\{0, 2\}, \{2, 4\}, \{4, 6\}, \{0, 6\}, \{1, 3\}, \{3, 5\}, \{5, 7\}, \{1, 7\}\}$. Thus $\Gamma(\mathbb{Z}_8, C^*)$ has two components, rather than one. $\square$

Since a group can have multiple Cayley graphs, we should at least partially investigate the conditions under which different Cayley graphs of the same group are isomorphic.

**Lemma 2.5** *Let $\theta$ be an automorphism of the group $G$. Then $\Gamma(G, C)$ and $\Gamma(G, \theta(C))$ are isomorphic.*

*Proof.* For any $x, y \in V(\Gamma(G,C)) = G$, we have

$$\theta(y)\theta(x)^{-1} = \theta(yx^{-1}),$$

which implies that $\theta(y)\theta(x)^{-1} \in \theta(C)$ if and only if $yx^{-1} \in C$. Therefore $\theta$ is an isomorphism from $\Gamma(G,C)$ to $\Gamma(G,\theta(C))$ [2]. ∎

## 2.2 Group Actions

We should show how Cayley graphs are indeed useful representations of groups. Before we continue, let us recall a couple of definitions related to group actions.

**Definition 2.6** A permutation group $S$ acting on a set $X$ is **transitive** if for every $x, y \in X$ there exists some $\sigma \in S$ such that $\sigma(x) = y$ Equivalently, we say that $S$ **acts transitively** on $X$[1]. ♢

**Definition 2.7** A permutation group $S$ acting on a set $X$ is **regular** if $S$ is transitive and no non-identity element of $S$ fixes any element of $X$. Equivalently, we say that $S$ **acts regularly** on $X$[2]. ♢

We now define vertex transitive graphs and show that Cayley graphs are vertex transitive.

**Definition 2.8** A graph $\Gamma$ is **vertex transitive** if the group of automorphisms of $\Gamma$, $\mathrm{Aut}(\Gamma)$, acts transitively on $\Gamma$, i.e. $\mathrm{Aut}(\Gamma)$ has only one orbit [3]. ♢

**Theorem 2.9** *The Cayley graph $\Gamma(G,C)$ is vertex transitive.*

*Proof.* Consider the right regular representation of $G$,

$$\rho_g : x \mapsto xg,$$

which is a permutation of the elements of $G$. Observe that

$$(yg)(xg)^{-1} = ygg^{-1}x^{-1} = yx^{-1},$$

so $\{xg, yg\} \in E(\Gamma(G,C))$ if and only if $\{x, y\} \in \Gamma(G,C)$. Hence $\rho_g$ is an automorphism of $\Gamma(G,C)$. By Cayley's theorem, the set of permutations $\overline{G} = \{\rho_g | g \in G\}$ forms a subgroup of $\mathrm{Aut}(\Gamma(G,C))$ isomorphic to $G$. Then for every $g, h \in G$, $\rho_{g^{-1}h}$ maps $g$ to $h$. Hence $\overline{G}$ acts transitively on $\Gamma(G,C)$, so $\Gamma(G,C)$ is vertex transitive [2]. ∎

**Remark 2.10** Though all Cayley graphs are vertex transitive, not all vertex transitive graphs are Cayley graphs. One example of such a graph is the Petersen graph [2].

In the proof above, we established that $\mathrm{Aut}(\Gamma(G,C))$ contains a subgroup $\overline{G}$ isomorphic to $G$ which acts transitively on $V(\Gamma) = G$. Since $\overline{G}$ is isomorphic to $G$, no non-identity element of $\overline{G}$ will fix any element of $V(\Gamma) = G$; thus $\overline{G}$ is regular, like $G$. We sum this up in the following lemma.

**Lemma 2.11** *Aut$(\Gamma(G,C))$ contains a regular subgroup isomorphic to $G$.* [2]

We now consider the converse of this result, which allows us to identify whether a given graph is a Cayley graph.

**Lemma 2.12** *If a group $G$ acts regularly on the vertices of a graph $\Gamma$, then $\Gamma$ is the Cayley graph of $G$ relative to some inverse-closed subset of $G \setminus e$.*

*Proof.* Grab a vertex $u$ of $\Gamma$. Since $G$ acts regularly on $V(\Gamma)$, we can define $g_v$ to be the element of $G$ such that $v = g_v(u)$. Define

$$C := \{g_v : v \text{ is adjacent to } u\}.$$

If $x$ and $y$ are vertices of $\Gamma$, then $g_x \in \mathrm{Aut}(\Gamma)$, and thus $x$ is adjacent to $y$ if and only if $g_x^{-1}(x)$ and $g_x^{-1}(y)$ are adjacent. However, $g_x^{-1}(x) = u$, and

$$g_x^{-1}(y) = g_y g_x^{-1}(u),$$

so $x$ and $y$ are adjacent if and only if $g_y g_x^{-1} \in C$. If we identify each vertex $x$ with $g_x$, then $\Gamma = \Gamma(G, C)$. The graph $\Gamma$ is undirected and has no loops, so $C$ is an inverse-closed subset of $G \setminus e$ [2]. ■

# 3 Cayley Graphs, Components, and Cosets

The structure of the Cayley graph $\Gamma(G, C)$ will depend on the subgroup of $G$ generated by $C$. Specifically, the Cayley graph gives a visual representation of the left cosets of the subgroup generated by $C$.

**Lemma 3.1** *Let $H$ be the subgroup of $G$ generated by an inverse-closed subset $C$ of $G \setminus e$. Then two vertices $u, v \in \Gamma(G, C)$ are in the same component of $\Gamma(G, C)$ if and only if $uH = vH$.*

*Proof.* First, assume that $u$ and $v$ are in the same component $\Gamma_k$ of $\Gamma(G, C)$. For any two vertices $u, v \in \Gamma_k$, there is at least one path from $u$ to $v$ in $\Gamma_k$, say $P = \{x_1, x_2, \ldots, x_{m-1}, x_m\}$, where $u = x_1$ and $v = x_m$. Then for $x_i, x_{i+1} \in P$, there is some $c_i \in C$ such that $x_{i+1} = c_i x_i$. Equivalently, $x_{i+1} x_i^{-1}$ is in $C$. Then $v = (v x_m^{-1})(x_m x_{m-1}^{-1}) \ldots (x_2 x_1^{-1})(x_1 u^{-1})u = hu$ for some $h \in H$. We may equivalently write that $h = vu^{-1}$, meaning $vu^{-1} \in H$. Thus by a well-known set of equivalences, $uH = vH$ [1].

Now, assume that $uH = vH$. Then $vu^{-1} \in H$, so $v = hu$ for some $h \in H$. We can write $h$ as the product of a word of elements of $C$, i.e. $h = c_m c_{m-1} \cdots c_2 c_1$ for (not necessarily distinct) $c_1, c_2, \ldots, c_m \in C$. Let $x_0 = u, x_1 = c_1 x_0, x_2 = c_2 x_1, \ldots, x_m = c_m x_{m-1}$. Notice that $x_m = v$, so there is a path $u, x_1, x_2, \ldots, x_{m-1}, v$ from $u$ to $v$. Hence, $u$ and $v$ are in the same component of $\Gamma(G, C)$. ■

This leads to the following corollary, which is presented as a lemma in [2].

**Corollary 3.2** *The Cayley graph $\Gamma(G, C)$ is connected if and only if $C$ is a generating set for $G$.*

*Proof.* If $\Gamma(G, C)$ is connected, then $\Gamma(G, C)$ has only one component. Hence $[G : \langle C \rangle] = 1$,, so $G = \langle C \rangle$.

If $C$ is a generating subset of $G$, then $[G : \langle C \rangle] = [G : G] = 1$. Then $\Gamma(G, C)$ has exactly only one component. ■

We can sum up the above results in the main theorem of this section.

**Theorem 3.3** *Let $H$ be the subgroup of $G$ generated by an inverse-closed subset $C$ of $G \setminus e$, and let $m = [G : H]$. Then the Cayley graph $\Gamma(G, C)$ has components $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$, where $V(\Gamma_1), V(\Gamma_2), \ldots, V(\Gamma_m)$ are the $m$ left cosets of $H$.*

*Proof.* By Lemma 3.1, any two elements $u, v \in G$ are in the same coset of $H$ if and only if they are in the same component of $\Gamma(G, C)$. There are exactly $m$ cosets of $H$, so it quickly follows that the cosets of $H$ are the vertex sets of the components of $\Gamma(G, C)$. ■

**Example 3.4** $\mathbb{Z}_8$ **Revisited.** Revisiting Example 2.3 and Example 2.4, notice that the number of components of $\Gamma(\mathbb{Z}_8, \{1, 7\})$ is $[\mathbb{Z}_8 : \langle \{1, 7\} \rangle] = [\mathbb{Z}_8 : \mathbb{Z}_8] = 1$, while the number of components of $\Gamma(\mathbb{Z}_8, \{2, 6\})$ is $[\mathbb{Z}_8 : \langle \{2, 6\} \rangle] = [\mathbb{Z}_8 : H] = 2$, where $H$ is the subgroup of $\mathbb{Z}_8$ isomorphic to $\mathbb{Z}_4$. □

# References


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