Cayley Graphs

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7 May 2019
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   - Components and Cosets
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Graph Theory Refresher

- **Graph**: a set of *vertices* and a set of *edges* between them.
- Directed vs. undirected graphs
- **Simple graph**: Undirected, unweighted edges; no loops; no multiple edges
- **Graph isomorphism**: Bijection $\phi : V(\Gamma) \rightarrow V(\Gamma')$ where
  \[
  \{u, v\} \in E(\Gamma) \iff \{\phi(u), \phi(v)\} \in E(\Gamma')
  \]
Cayley Graphs and Group Actions

Introducing Cayley Graphs
Cayley Graphs

Definition

$G$ group, and $C$ inverse-closed subset of $G$. The Cayley graph of $G$ relative to $C$, $\Gamma(G, C)$, is a simple graph defined as follows:

- $V(\Gamma) = G$
- $E(\Gamma) = \{\{g, h\} | hg^{-1} \in C\}$.

That is, $\{g, h\} \in E(\Gamma)$ if and only if there is some $c \in C$ such that $h = cg = \lambda_c(g)$.

Note: we call $C$ the connection set of $\Gamma(G, C)$. 

Introducing Cayley Graphs

One Group, Different Cayley Graphs

Example ($\mathbb{Z}_8$, $C$ generates $\mathbb{Z}_8$)

$C = \{1, -1\} = \{1, 7\}$
One Group, Different Cayley Graphs

Example ($\mathbb{Z}_8$, $C$ generates subgroup $\cong \mathbb{Z}_4$)

$C = \{2, 6\}$

![Cayley Graph Diagram](image-url)
One Cayley Graph, Two Different Groups

Example \((G = S_3, C = \{(123), (132), (12)\})\)
Introducing Cayley Graphs

One Cayley Graph, Two Different Groups

Example \((G = \mathbb{Z}_6, C = \{2, 4, 3\})\)
A Note about Definitions

There are different ways to define Cayley graphs.

- **Connected** Cayley graphs: these require that $C$ be a generating set for $G$.

- **Directed** Cayley graphs: these do not require $C$ to be inverse-closed.

- **Colored, directed** Cayley graphs: edges $(g, h)$ are colored/labeled based on which $c \in C$ satisfies $h = cg$.

Notice: () vs {} for undirected vs. directed edges
Lemma

Let $\theta$ be an automorphism of $G$. Then $\Gamma(G, C) \cong \Gamma(G, \theta(C))$.

Proof.

For any $x, y \in G$,

$$\theta(y)\theta(x)^{-1} = \theta(yx^{-1}),$$

so $\theta(y)\theta(x)^{-1} \in C$ if and only if $yx^{-1} \in C$. Hence $\theta$ is an isomorphism from $\Gamma(G, C)$ to $\Gamma(G, \theta(C))$. $\square$
Group Actions and Vertex Transitivity
Theorem (Cayley)

Every group is isomorphic to a group of permutations.

Proof idea.

Consider the left regular representation \( \lambda_g : G \to G \), defined by

\[ \lambda_g(x) = gx. \]

Note: We could have instead considered the right regular representation \( \rho_g : G \to G \), defined as \( \rho_g(x) = xg \).
Let $S$ be a permutation group acting on a set $X$.

**Definition**

$S$ is **transitive** if for every $x, y \in X$, there is $\sigma \in S$ such that $\sigma(x) = y$.

**Definition**

$S$ is **regular** if it is transitive and the only $\sigma \in S$ that fixes any element of $X$ is the identity.

We say $S$ acts **transitively/regularly** (resp.) on $X$. 
A graph $\Gamma$ is **vertex transitive** if $\text{Aut}(G)$ acts transitively on $\Gamma$, i.e. $\text{Aut}(G)$ has only one orbit.

**Example (Not vertex transitive)**

Also not regular.
The Cayley graph $\Gamma(G,C)$ is vertex transitive.

Proof.
Consider the right regular representation of $G$, $\rho_g : x \mapsto xg$.
Observe that

$$(yg)(xg)^{-1} = ygg^{-1}x^{-1} = yx^{-1},$$

so $\{xg, yg\} \in E(\Gamma(G,C))$ if and only if $\{x, y\} \in E(\Gamma(G,C))$.

Then $\rho_g$ is an automorphism of $\Gamma(G,C)$. By Cayley’s Theorem, $\overline{G} = \{\rho_g | g \in G\}$ forms a subgroup of $\text{Aut}(\Gamma(G,C))$ isomorphic to $G$. For $g, h \in G$, $\rho_{g^{-1}}h(g) = h$. Thus $\overline{G}$ acts transitively on $\Gamma(G,C)$. 

\qed
Corollary

\text{Aut}(\Gamma(G,C)) \text{ has a regular subgroup isomorphic to } G.

Proof.

\bar{G} = \{\rho_g | g \in G\} \text{ is a subgroup of } \text{Aut}(\Gamma(G,C)) \text{ that acts transitivity on } V(\Gamma) = G. \text{ Since } \bar{G} \cong G, \text{ only the identity will fix any element of } V(\Gamma) = G. \text{ Thus } \bar{G} \text{ is regular.}
A Way to Identify Cayley Graphs

Theorem

If a group $G$ acts regularly on the vertices of $\Gamma$, then $\Gamma$ is the Cayley graph of $G$ relative to some inverse-closed $C \subset G \setminus e$.

Proof.

Grab $u \in V(\Gamma)$. Let $g_v$ be the element of $G$ such that $v = g_v(u)$. Define $C := \{g_v : v \text{ is adjacent to } u\}$.

If $x, y \in V(\Gamma)$, then $g_x \in \text{Aut}(\Gamma)$, so $x \sim y$ if and only if $g_x^{-1}(x) \sim g_x^{-1}(y)$. But $g_x^{-1}(x) = u$, and $g_x^{-1}(y) = g_y g_x^{-1}(u)$, so $x \sim y$ if and only if $g_y g_x^{-1} \in C$.

Identify each vertex $x$ with $g_x$. Then $\Gamma = \Gamma(G, C)$. $\Gamma$ is undirected with no loops, so $C$ is an inverse-closed subset of $G \setminus e$. 

Not all vertex-transitive graphs are Cayley graphs. Example: the Petersen graph.

Example (Petersen graph)

Only two groups of order 10: $\mathbb{Z}_{10}$ and $D_5$. 
Structure of the Cayley graph

How to anticipate the structure of the Cayley graph $\Gamma(G,C)$?

- Examine the subgroup generated by $C$.
- The Cayley graph gives a visual representation of the left cosets of the subgroup generated by $C$.

Time to examine the components of a Cayley graph...
Components of the Cayley graph

Lemma (Same Coset, Same Component)

Let $H$ be the subgroup of $G$ generated by an inverse-closed subset $C$ of $G \setminus e$. Then two vertices $u, v$ in $\Gamma(G, C)$ are in the same component of $\Gamma(G, C)$ if and only if $uH = vH$.

Proof. ($\Rightarrow$).

Assume $u, v$ in the same component $\Gamma_k$ of $\Gamma(G, C)$. Then there is at least one path from $u$ to $v$, $P = \{x_1, x_2, \ldots, x_m\}$, where $x_1 = u$ and $x_m = v$. So $x_{i+1}x_i^{-1} \in C$ for $1 \leq i < m$. Then

$$v = (vx_{m-1}^{-1})(x_{m-1}x_{m-2}^{-1})\cdots(x_2u^{-1})u = hu$$

, for some $h \in H$. Equivalently, $h = vu^{-1}$, so $vu^{-1} \in H$. Then $uH = vH$. 

Components of the Cayley graph

Proof. (⇐).

Assume $uH = vH$. Then $vu^{-1} \in H$, so $v = hu$ for some $h \in H$. Further, $h = c_mc_{m-1} \cdots c_2c_1$ where $c_i \in C$, $1 \leq i \leq m$.

Let $x_0 = u, x_1 = c_1x_0, x_2 = c_2x_1, \ldots, x_m = c_mx_{m-1} = v$. Then we have a path from $u$ to $v$, namely, $P = \{u, x_1, x_2, \ldots, x_{m-1}, v\}$. Thus $u$ and $v$ are in the same component of $\Gamma(G, C)$. 

□
When are Cayley graphs connected?

**Corollary**

The Cayley graph $\Gamma(G, C)$ is connected if and only if $C$ generates $G$.

**Proof.**

If $\Gamma(G, C)$ is connected, then it has only one component. Hence $[G : \langle C \rangle] = 1$, so $G = \langle C \rangle$.

If $C$ generates $G$, then $[G : \langle C \rangle] = [G : G] = 1$, so $\Gamma(G, C)$ has exactly one component.
Theorem (Cosets As Components)

Let $H$ be the subgroup of $G$ generated by an inverse-closed subset $C$ of $G \setminus e$, and let $m = [G : H]$. Then the Cayley graph $\Gamma(G, C)$ has components $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, where $V(\Gamma_1), V(\Gamma_2), \ldots, V(\Gamma_m)$ are the $m$ left cosets of $H$ in $G$.

Proof.

By Lemma SCSC, any two elements $u, v \in G$ are in the same coset of $H$ if and only if the are in the same component of $\Gamma(G, C)$. $[G : H] = m$, so the cosets of $H$ in $G$ are the vertex sets of the components of $\Gamma(G, C)$. \qed
Revisiting $\mathbb{Z}_8$

Example (In Light of Cosets As Components)

[Diagram showing a cycle graph for $\mathbb{Z}_8$ and two other graphs illustrating cosets]
Direct Products and Cayley Graphs
**Fun with $\mathbb{Z}_{10}$**

$\mathbb{Z}_{10}$'s nontrivial proper subgroups

- $H = \langle 2 \rangle \cong \mathbb{Z}_5$
- $K = \langle 5 \rangle \cong \mathbb{Z}_2$

**Example**

![Cayley Graph](image-url)
An interesting Cayley graph

\( \mathbb{Z}_{10} \) is the inner direct product of \( \langle 5 \rangle \) and \( \langle 2 \rangle \), and thus \( \mathbb{Z}_{10} \cong \langle 5 \rangle \times \langle 2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_5 \).

Example (\( G = \mathbb{Z}_{10}, C = \{2, 8\} \cup \{5\} = \{2, 5, 8\} \))

![Cayley Graph Diagram]
Cartesian Product of Graphs

Definition

Given two graphs $X$ and $Y$, we define their **Cartesian product**, $X \square Y$, as having vertex set $V(X) \times V(Y)$, where $\{(x_1, y_1), (x_2, y_2)\} \in E(X \square Y)$ if and only if one of the following conditions is met:

- $x_1 = x_2$ and $y_1 \sim y_2$
- $y_1 = y_2$ and $x_1 \sim x_2$
Thank You!