Cayley Graphs

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7 May 2019

Overview

Graph Theory Refresher

- **Graph**: a set of *vertices* and a set of *edges* between them.
- Directed vs. undirected graphs
- **Simple graph**: Undirected, unweighted edges; no loops; no multiple edges
- **Graph isomorphism**: Bijection $\phi : V(\Gamma) \to V(\Gamma')$ where

$$\{u, v\} \in E(\Gamma) \iff \{\phi(u), \phi(v)\} \in E(\Gamma')$$

Cayley Graphs and Group Actions

Cayley Graphs

Definition

G group, and C inverse-closed subset of G. The **Cayley graph** of G relative to C, $\Gamma(G, C)$, is a simple graph defined as follows:

- $V(\Gamma) = G$
- $E(\Gamma) = \{\{g, h\} | hg^{-1} \in C\}$.

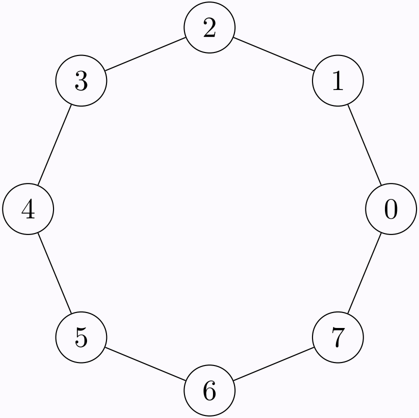
That is, $\{g, h\} \in E(\Gamma)$ if and only if there is some $c \in C$ such that $h = cg = \lambda_c(g)$.

Note: we call C the **connection set** of $\Gamma(G, C)$.

One Group, Different Cayley Graphs

Example ($\mathbb{Z}_8$, C generates $\mathbb{Z}_8$)

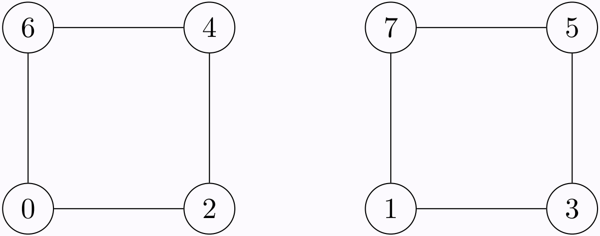
$C = \{1, -1\} = \{1, 7\}$



One Group, Different Cayley Graphs

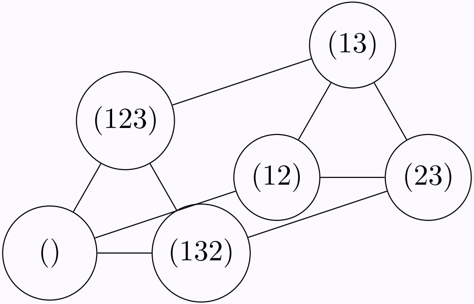
Example ($\mathbb{Z}_8$, C generates subgroup $\cong \mathbb{Z}_4$)

$C = \{2, 6\}$



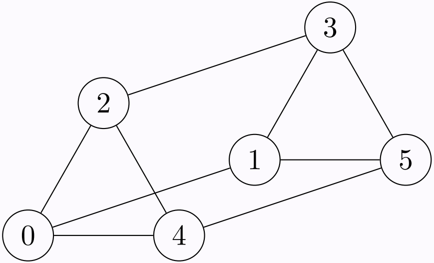
One Cayley Graph, Two Different Groups

Example ($G = S_3, C = \{(123), (132), (12)\}$)



One Cayley Graph, Two Different Groups

Example ($G = \mathbb{Z}_6, C = \{2, 4, 3\}$)



A Note about Definitions

There are different ways to define Cayley graphs.

- *Connected* Cayley graphs: these require that C be a generating set for G.
- *Directed* Cayley graphs: these do not require C to be inverse-closed.
- *Colored, directed* Cayley graphs: edges (g, h) are colored/labeled based on which $c \in C$ satisfies $h = cg$.

Notice: () vs {} for undirected vs. directed edges

Lemma

Let θ be an automorphism of G. Then $\Gamma(G, C) \cong \Gamma(G, \theta(C))$.

Proof.

For any $x, y \in G$,

$$\theta(y)\theta(x)^{-1} = \theta(yx^{-1}),$$

so $\theta(y)\theta(x)^{-1} \in C$ if and only if $yx^{-1} \in C$. Hence θ is an isomorphism from $\Gamma(G, C)$ to $\Gamma(G, \theta(C))$. □

Group Actions and Vertex Transitivity

Cayley's Theorem

Theorem (Cayley)

Every group is isomorphic to a group of permutations.

Proof idea.

Consider the *left regular representation* $\lambda_g : G \to G$, defined by

$$\lambda_g(x) = gx.$$

□

Note: We could have instead considered the *right regular representation* $\rho_g : G \to G$, defined as $\rho_g(x) = xg$.

Transitive and Regular Group Actions

Let S be a permutation group acting on a set X.

Definition

S is **transitive** if for every $x, y \in X$, there is $\sigma \in S$ such that $\sigma(x) = y$.

Definition

S is **regular** if it is transitive and the only $\sigma \in S$ that fixes *any* element of X is the identity.

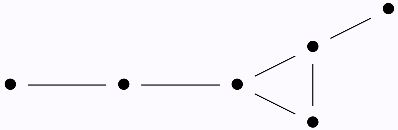
We say S **acts transitively/regularly** (resp.) on X.

Vertex Transitive Graphs

Definition

A graph Γ is **vertex transitive** if $\mathrm{Aut}(G)$ acts transitively on Γ, i.e. $\mathrm{Aut}(G)$ has only one orbit.

Example (Not vertex transitive)



Also not regular.

Vertex Transitive Graphs

Theorem

The Cayley graph $\Gamma(G, C)$ *is vertex transitive.*

Proof.

Consider the *right regular representation* of G, $\rho_g : x \mapsto xg$. Observe that

$$(yg)(xg)^{-1} = ygg^{-1}x^{-1} = yx^{-1},$$

so $\{xg, yg\} \in E(\Gamma(G, C))$ if and only if $\{x, y\} \in E(\Gamma(G, C))$. Then ρ_g is an automorphism of $\Gamma(G, C)$. By Cayley's Theorem, $\overline{G} = \{\rho_g | g \in G\}$ forms a subgroup of $\mathrm{Aut}(\Gamma(G, C))$ isomorphic to G. For $g, h \in G$, $\rho_{g^{-1}h}(g) = h$. Thus $\overline{G}$ acts transitively on $\Gamma(G, C)$. □

Corollary

$\mathrm{Aut}(\Gamma(G, C))$ *has a regular subgroup isomorphic to* G*.*

Proof.

$\overline{G} = \{\rho_g | g \in G\}$ is a subgroup of $\mathrm{Aut}(\Gamma(G, C))$ that acts transitively on $V(\Gamma) = G$. Since $\overline{G} \cong G$, only the identity will fix any element of $V(\Gamma) = G$. Thus $\overline{G}$ is regular. □

A Way to Identify Cayley Graphs

Theorem

If a group G acts regularly on the vertices of Γ, then Γ is the Cayley graph of G relative to some inverse-closed $C \subset G \setminus e$.

Proof.

Grab $u \in V(\Gamma)$. Let g_v be the element of G such that $v = g_v(u)$. Define $C := \{g_v : v \text{ is adjacent to } u\}$.

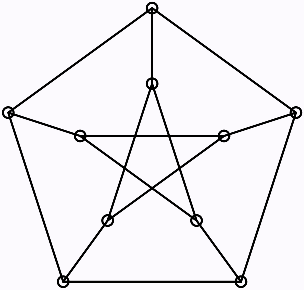
If $x, y \in V(\Gamma)$, then $g_x \in \mathrm{Aut}(\Gamma)$, so $x \sim y$ if and only if $g_x^{-1}(x) \sim g_x^{-1}(y)$. But $g_x^{-1}(x) = u$, and $g_x^{-1}(y) = g_y g_x^{-1}(u)$, so $x \sim y$ if and only if $g_y g_x^{-1} \in C$.

Identify each vertex x with g_x. Then $\Gamma = \Gamma(G, C)$. Γ is undirected with no loops, so C is an inverse-closed subset of $G \setminus e$. □

Remark

Not all vertex-transitive graphs are Cayley graphs. Example: the Petersen graph.

Example (Petersen graph)



Only two groups of order 10: $\mathbb{Z}_{10}$ and D_5.

Structure of the Cayley graph

How to anticipate the structure of the Cayley graph $\Gamma(G, C)$?

- Examine the subgroup generated by C.
- The Cayley graph gives a visual representation of the *left cosets* of the subgroup generated by C.

Time to examine the components of a Cayley graph...

Components of the Cayley graph

Lemma (Same Coset, Same Component)

Let H be the subgroup of G generated by an inverse-closed subset C of $G \setminus e$. Then two vertices u, v in $\Gamma(G, C)$ are in the same component of $\Gamma(G, C)$ if and only if $uH = vH$.

Proof. ($\Rightarrow$).

Assume u, v in the same component Γ_k of $\Gamma(G, C)$. Then there is at least one path from u to v, $P = \{x_1, x_2, \ldots, x_m\}$, where $x_1 = u$ and $x_m = v$. So $x_{i+1}x_i^{-1} \in C$ for $1 \leq i < m$. Then

$$v = (vx_{m-1}^{-1})(x_{m-1}x_{m-2}^{-1}) \cdots (x_2u^{-1})u = hu$$

, for some $h \in H$. Equivalently, $h = vu^{-1}$, so $vu^{-1} \in H$. Then $uH = vH$. □

Components of the Cayley graph

Proof. ($\Leftarrow$).

Assume $uH = vH$. Then $vu^{-1} \in H$, so $v = hu$ for some $h \in H$. Further, $h = c_m c_{m-1} \cdots c_2 c_1$ where $c_i \in C$, $1 \leq i \leq m$.

Let $x_0 = u, x_1 = c_1 x_0, x_2 = c_2 x_1, \ldots, x_m = c_m x_{m-1} = v$. Then we have a path from u to v, namely,
$P = \{u, x_1, x_2, \ldots, x_{m-1}, v\}$. Thus u and v are in the same component of $\Gamma(G, C)$. $\square$

When are Cayley graphs connected?

Corollary

The Cayley graph $\Gamma(G, C)$ *is connected if and only if* C *generates* G*.*

Proof.

If $\Gamma(G, C)$ is connected, then it has only one component. Hence $[G : \langle C \rangle] = 1$, so $G = \langle C \rangle$.

If C generates G, then $[G : \langle C \rangle] = [G : G] = 1$, so $\Gamma(G, C)$ has exactly one component. □

Theorem (Cosets As Components)

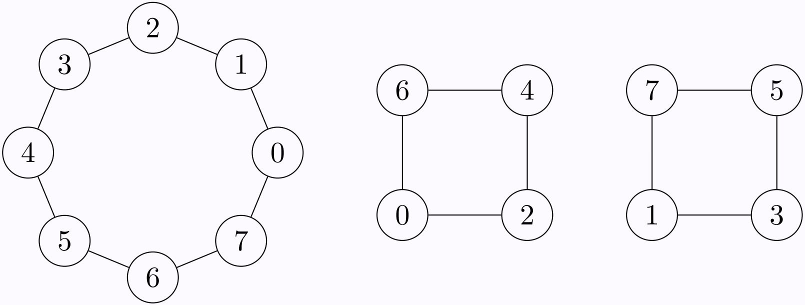
Let H be the subgroup of G generated by an inverse-closed subset C of $G \setminus e$, and let $m = [G : H]$. Then the Cayley graph $\Gamma(G, C)$ has components $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, where $V(\Gamma_1), V(\Gamma_2), \ldots, V(\Gamma_m)$ are the m left cosets of H in G.

Proof.

By Lemma SCSC, any two elements $u, v \in G$ are in the same coset of H if and only if the are in the same component of $\Gamma(G, C)$. $[G : H] = m$, so the cosets of H in G are the vertex sets of the components of $\Gamma(G, C)$. □

Revisiting $\mathbb{Z}_8$

Example (In Light of Cosets As Components)

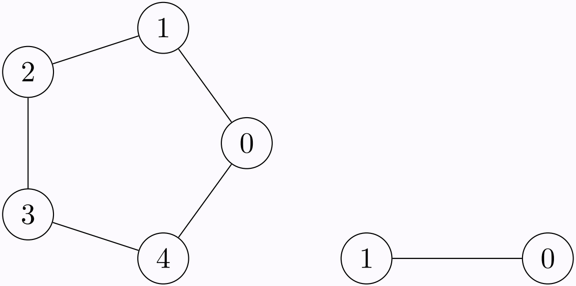


Direct Products and Cayley Graphs

$\mathbb{Z}_{10}$'s nontrivial proper subgroups

- $H = \langle 2 \rangle \cong \mathbb{Z}_5$
- $K = \langle 5 \rangle \cong \mathbb{Z}_2$

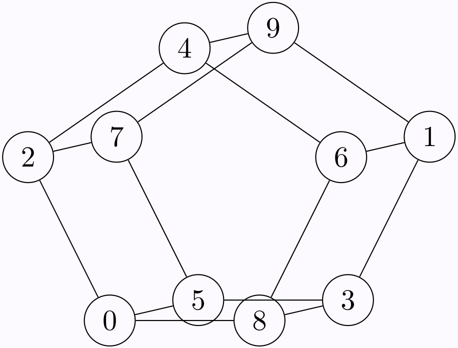
Example



An interesting Cayley graph

$\mathbb{Z}_{10}$ is the inner direct product of $\langle 5 \rangle$ and $\langle 2 \rangle$, and thus $\mathbb{Z}_{10} \cong \langle 5 \rangle \times \langle 2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_5$.

Example ($G = \mathbb{Z}_{10}$, $C = \{2, 8\} \cup \{5\} = \{2, 5, 8\}$)



Cartesian Product of Graphs

Definition

Given two graphs X and Y, we define their **Cartesian product**, $X \square Y$, as having vertex set $V(X) \times V(Y)$, where $\{(x_1, y_1), (x_2, y_2)\} \in E(X \square Y)$ if and only if one of the following conditions is met:

- $x_1 = x_2$ and $y_1 \sim y_2$
- $y_1 = y_2$ and $x_1 \sim x_2$

Thank You!