Universal Algebra

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Motivation

- "A group is defined to consist of a nonempty set G together with a binary operation $\circ$ satisfying the axioms."
- "A field is defined to consist of a nonempty set F together with two binary operations $+$ and $\cdot$ satisfying the axioms ..."
- "A vector space is defined to consist of a nonempty set V together with a binary operation $+$ and, for each number r, an operation called scalar multiplication such that ..."

Motivation

How can we generalize the different structures we encounter in an abstract algebra course?

Some definitions

Definition

Given an equivalence relation θ on A, the EQUIVALENCE CLASS of $a \in A$ is the set

$$a/\theta = \{b \in A \mid \langle a, b \rangle \in \theta\}.$$

Definition

The QUOTIENT SET OF A BY θ is the set

$$A/\theta = \{a/\theta \mid a \in A\}$$

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Given an equivalence relation θ on A, the CANONICAL MAP is the function $\phi : A \to A/\theta$ where

$$\phi(a) = a/\theta.$$

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Nothing out of the ordinary so far.

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Pros of this new definition

- Assumes nothing about any structure on A or B.
- The kernel is an *equivalence relation*

The First Theorem

Theorem

If $\psi : A \to B$ is a function with $K = \ker(\psi)$, then K is an equivalence relation on A. Let $\phi : A \to A/K$ be the canonical map. Then there exists a unique bijection $\eta : A/K \to \psi(A)$ such that $\psi = \eta\phi$.

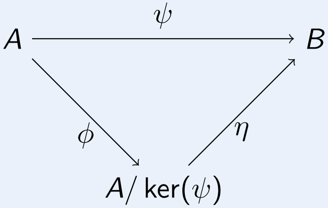

Figure: *Commutative Diagram*

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Example

If f is a binary operation on $A = \{a, b, c\}$.

f	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

Algebras

Definition

A SIGNATURE $\mathscr{F}$ is a set of function *symbols*. Each symbol $f \in \mathscr{F}$ is assigned an integer called its ARITY.

- In Universal algebra signatures are sometimes called *types*.
- Sometimes signatures are defined only in terms of their arities.

Algebras

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An ALGEBRA **A** with UNIVERSE A and signature $\mathscr{F}$ is a pair $\langle A, F \rangle$, where F is a set of functions corresponding to symbols in $\mathscr{F}$.

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We distinguish between function *symbols* and the "actualy" function with a superscript.
For example $f \in \mathscr{F}$, and $f^{\mathbf{A}} \in F$.

Example of an Algebra

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$+^{\mathbb{Z}_3}$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$()^{-1\ \mathbb{Z}_3}$	0	1	2
	0	2	1

$\mathbf{1}^{\mathbb{Z}_3} : \emptyset \mapsto 0$

Congruences

A CONGRUENCE is a special kind of equivalence relation. They are the equivalence relations which "respect" the operations of an algebra.

$$a_1 \sim b_1, a_2 \sim b_2 \implies f^{\mathbf{A}}(a_1, a_2) \sim f^{\mathbf{A}}(b_1, b_2)$$

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Example

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Consider the subgroup $\{0, 4, 8\}$ of $\mathbb{Z}_{12}$. Define the relation $a \sim b$ to mean a and b are in the same coset.
Then we have

$$a_1 = 0 \sim 8 = b_1$$
$$a_2 = 7 \sim 3 = b_2$$
$$f^{\mathbf{A}}(a_1, a_2) = (0 + 7) \sim (8 + 3) = f^{\mathbf{A}}(b_1, b_2)$$

The congruence relation is preserved under the operation $+$

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Definition

If $\mathbf{A}$ and $\mathbf{B}$ are two algebras with the same signature $\mathscr{F}$, then $\varphi : \mathbf{A} \to \mathbf{B}$ is a homomorphism if for every n-ary function symbol $f \in \mathscr{F}$ and every $a_1, \ldots, a_n \in A$,

$$\varphi(f^{\mathbf{A}}(a_1, \ldots, a_n)) = f^{\mathbf{B}}(\varphi(a_1), \ldots, \varphi(a_n)).$$

We can "push φ through" operations.

Congruences and Homomorphisms

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(they were secretly defined that way) □

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If θ is a congruence on $\mathbf{A}$, then the quotient algebra of $\mathbf{A}$ by θ is the algebra $\mathbf{A}/\theta$ with the same signature as $\mathbf{A}$, and whose universe is A/θ.

Example

Returning to the subgroup $H = \{0, 4, 8\}$ of $\mathbb{Z}_{12}$ and the congruence $\sim$. We can define the quotient algebra $\mathbb{Z}_{12}/\sim$.

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and in group theory we write

$$(0+H) + (3+H) = (0+3) + H.$$

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Corollary

If $\varphi : \mathbf{A} \to \mathbf{B}$ *is a homomorphism, then* $\mathbf{A}/\ker(\varphi)$ *is a quotient algebra.*

The First Isomorphism Theorem

Theorem

(The First Isomorphism Theorem) If $\psi : \mathbf{A} \to \mathbf{B}$ *is a homomorphism with* $K = \ker(\psi)$*, then* K *is a congruence on* A*. Let* $\phi : \mathbf{A} \to \mathbf{A}/\ker(\psi)$ *be the canonical homomorphism. Then there exists a unique isomorphism* $\eta : \mathbf{A}/\ker(\psi) \to \psi(\mathbf{A})$ *such that* $\psi = \eta \circ \phi$*.*

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Exactly the same as before, but now our bijection is an isomorphism!

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The set of all congruences on an algebra forms a lattice with joins $\bigvee = \bigcup$ and meets $\bigwedge = \bigcap$.

This lattice has a top and bottom as well.
Top is the relation

$$A^2$$

and bottom is

$$\{\langle a, a\rangle \mid a \in A\}.$$

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A Poset **P** is a relational structure with signature $\mathscr{R} = \{\preccurlyeq\}$.

First-Order structures

What if we allow both relations an operations in our signature?

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