Math 290 Name: Key Dr. Beezer
Exam 7 Spring 2019
Chapter R

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate matrices, row-reduce matrices, and compute eigenvalues. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. E is a basis for $\mathbb{C}^3$, and F is a basis for P_1, the vector space of polynomials with degree at most 1. Find the matrix representation of T relative to E and F. (20 points)

$$T\colon \mathbb{C}^3 \to P_1, \quad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (a-b+2c)+(3a+b+c)\,x; \qquad E = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}\right\} \qquad F = \{1+x,\ 3+4x\}$$

$$\rho_F\left(T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)\right) = \rho_F(0+4x) = \rho_F(-12(1+x)+4(3+4x)) = \begin{bmatrix} -12 \\ 4 \end{bmatrix}$$

$$\rho_F\left(T\left(\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix}\right)\right) = \rho_F(7+17x) = \rho_F(-23(1+x)+10(3+4x)) = \begin{bmatrix} -23 \\ 10 \end{bmatrix}$$

$$\rho_F\left(T\left(\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}\right)\right) = \rho_F(9+18x) = \rho_F(-18(1+x)+9(3+4x)) = \begin{bmatrix} -18 \\ 9 \end{bmatrix}$$

$$M^T_{E,F} = \begin{bmatrix} -12 & -23 & -18 \\ 4 & 10 & 9 \end{bmatrix}$$

2. Using the linear transformation T from the previous problem, compute $T\left(\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}\right)$. Then repeat the computation using your matrix representation above and the Fundamental Theorem of Matrix Representation, explaining clearly how you are applying the theorem. (15 points)

$T\left(\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}\right) = -2+7x$, directly. Now w/ FTMR

① Coordinatize input: $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 4 \\ 4 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ vector rep $= \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

② (matrix rep) $\cdot$ (vector rep) $= \begin{bmatrix} -12 & -23 & -18 \\ 4 & 10 & 9 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -29 \\ 9 \end{bmatrix}$

③ Un-coordinatize output: $-29(1+x)+9(3+4x) = -2+7x$

3. Consider the linear transformation S below. (P_2 is the vector space of polynomials with degree at most 2.) Find a basis for P_2 so that the matrix rerpresentation of S is a diagonal matrix. (15 points)

$$S\colon P_2 \to P_2, \qquad S\left(a+bx+cx^2\right) = (12a+15b-15c) + (-5a-8b+5c)\,x + (5a+5b-8c)\,x^2$$

Build a simple matrix representation w/ basis $B=\{1, x, x^2\}$

$$M^S_{B,B} = \begin{bmatrix} 12 & 15 & -15 \\ -5 & -8 & 5 \\ 5 & 5 & -8 \end{bmatrix}$$

Eigenvalues? Eigenvectors? Sage: $\lambda = 2, -3, -3$

$\lambda=2$ $\quad M-2I_3 \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{x}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

$\lambda=-3$ $\quad M-(-3I_3) \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

algebraic multiplicities equal geometric multiplicities So we know these 3 vectors are linearly ind.

$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is the desired basis of P_2.

4. The linear transformation T below is invertible (you can assume this). Determine a formula for the inverse linear transformation T^{-1} by computing the matrix inverse of a matrix linear transformation. (P_2 is the vector space of polynomials with degree at most 2.) (15 points)

$$T\colon P_2 \to \mathbb{C}^3, \qquad T\left(a+bx+cx^2\right) = \begin{bmatrix} a+2b+7c \\ b+3c \\ a+b+5c \end{bmatrix}$$

use nice bases & build invertible matrix representation

$B = \{1, x, x^2\}$ $\quad C = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ $\quad M^T_{B,C} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{bmatrix}$

$$M^{T^{-1}}_{C,B} = \left(M^T_{B,C}\right)^{-1} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 3 \\ 1 & 1 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 1 \end{bmatrix}$$

$$T^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \rho_B^{-1}\left(M^{T^{-1}}_{C,B}\, \rho_C\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)\right) = \rho_B^{-1}\left(\begin{bmatrix} 2 & -3 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$$

$$= \rho_B^{-1}\left(\begin{bmatrix} 2a-3b-c \\ 3a-2b-3c \\ -a+b+c \end{bmatrix}\right)$$

$$= (2a-3b-c) + (3a-2b-3c)\,x + (-a+b+c)\,x^2$$

Compare with the 25-point problem on Exam LT.

5. Repeat the first problem of this examination, but begin by building a matrix representation of T relative to the nicest, simplest possible bases you can imagine. Then use change-of-basis matrices to convert the current representation into your previous one. (The relevant information from the first problem is duplicated here for convenience.) (20 points)

$$T\colon \mathbb{C}^3 \to P_1,\quad T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = (a-b+2c)+(3a+b+c)\,x; \qquad E=\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}3\\4\\4\end{bmatrix},\begin{bmatrix}3\\4\\5\end{bmatrix}\right\} \qquad F=\{1+x,\ 3+4x\}$$

Nice bases: $A = \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$, $B = \{1, x\}$

EZ matrix representation, $M^T_{A,B} = \begin{bmatrix}1 & -1 & 2\\ 3 & 1 & 1\end{bmatrix}$

$$M^T_{E,F} = C_{B,F}\, M^T_{A,B}\, C_{E,A} = C^{-1}_{F,B}\, M^T_{A,B}\, C_{E,A}$$

$$= \begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1} \begin{bmatrix}1 & -1 & 2\\ 3 & 1 & 1\end{bmatrix} \begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 4\\ 0 & 4 & 5\end{bmatrix}$$

$$= \begin{bmatrix}-12 & -23 & -18\\ 4 & 10 & 9\end{bmatrix}$$

6. Suppose that $T\colon U \to V$ is an invertible linear transformation. Prove that the matrix representation of T relative to bases B and C of U and V (respectively) is a nonsingular matrix. Important: construct a proof that uses the definition of a matrix representation from Chapter R, but otherwise mostly depends on results from the previous chapter (Chapter LT). In other words, so not simply claim that the matrix rerpresentation of an inverse linear transformation is a matrix inverse. (15 points)

① T invertible, ρ_C invertible $\Rightarrow$ $\rho_C \circ T$ invertible (Theorem CIVLT)

② Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ basis of U. Then $S = \{\rho_C(T(\mathbf{u}_1)), \rho_C(T(\mathbf{u}_2)), \ldots, \rho_C(T(\mathbf{u}_n))\}$ is the set of columns of the matrix representation $M^T_{B,C}$ (Definition MR).

③ Theorem ILTB $\Rightarrow$ S is a linearly independent set of column vectors.

④ Theorem NMLIC $\Rightarrow$ matrix representation is non singular.