Math 290
Exam 6
Chapter LT

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Verify that the function T is a linear transformation. (15 points)

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \qquad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b \\ b-c \end{bmatrix}$$

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = T\left(\begin{bmatrix} a+x \\ b+y \\ c+z \end{bmatrix}\right) = \begin{bmatrix} a+x+b+y \\ b+y-(c+z) \end{bmatrix} = \begin{bmatrix} a+b \\ b-c \end{bmatrix} + \begin{bmatrix} x+y \\ y-z \end{bmatrix} = T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) + T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

$$T\left(\alpha \begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = T\left(\begin{bmatrix} \alpha a \\ \alpha b \\ \alpha c \end{bmatrix}\right) = \begin{bmatrix} \alpha a + \alpha b \\ \alpha b - \alpha c \end{bmatrix} = \begin{bmatrix} \alpha(a+b) \\ \alpha(b-c) \end{bmatrix} = \alpha \begin{bmatrix} a+b \\ b-c \end{bmatrix} = \alpha T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)$$

2. Answer the following yes/no questions about T. Full credit requires a complete and convincing explanation, a simple "yes" or "no" will get no credit. (P_1 is the vector space of polynomials with degree at most 1, and M_{22} is teh vector space of 2×2 matrices.) (15 points)

$$T\colon P_1 \to M_{22}, \qquad T(a+bx) = \begin{bmatrix} a+2b & 2a+b \\ -a+b & a+2b \end{bmatrix}$$

homogeneous system w/ coefficient matrix

(a) Is T injective? $\mathcal{K}(T) = ?$

$T(a+bx) = \mathbf{0}$

$$\begin{bmatrix} a+2b & 2a+b \\ -a+b & a+2b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} a=b=0 \\ \mathcal{K}(T) = \{\mathbf{0}\} \end{matrix}$$

Theorem KILT $\Rightarrow$ T is injective, "yes"

(b) Is T surjective?

$2 = \dim P_1 \not\geq \dim M_{22} = 4$
T surjective would violate Theorem SLTD. So "no".

(c) Is T invertible?

"no" Theorem ILTIS requires T to be both injective & surjective.

3. The linear transformation T below is invertible (you can assume this). Determine a formula for the inverse linear transformation T^{-1}. (P_2 is the vector space of polynomials with degree at most 2.) (25 points)

$$T\colon P_2 \to \mathbb{C}^3, \qquad T\left(a + bx + cx^2\right) = \begin{bmatrix} a + 2b + 7c \\ b + 3c \\ a + b + 5c \end{bmatrix}$$

Compute (singleton) pre-images on a basis of $\mathbb{C}^3$: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

$T\{(a+bx+cx^2)\} = \begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow$ system $\left[\begin{array}{ccc|c} 1 & 2 & 7 & 1 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 5 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$ $T^{-1}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \{2+3x-x^2\}$

$T(a+bx+cx^2) = \begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow$ system $\left[\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 0 & 1 & 3 & 1 \\ 1 & 1 & 5 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$ $T^{-1}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \{-3-2x+x^2\}$

$T(a+bx+cx^2) = \begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow$ system $\left[\begin{array}{ccc|c} 1 & 2 & 7 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 5 & 1 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$ $T^{-1}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \{-1-3x+x^2\}$

So

$$T^{-1}\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = T^{-1}\left(a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = aT^{-1}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + bT^{-1}\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) + cT^{-1}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

$$= a(2+3x-x^2) + b(-3-2x+x^2) + c(-1-3x+x^2)$$

$$= (2a-3b-c) + (3a-2b-3c)x + (-a+b+c)x^2$$

4. Since the linear transformation T in the previous question is invertible, we know that P_2 and $\mathbb{C}^3$ are isomorphic vector spaces. Compute the sum of $2 + 3x - 5x^2$ and $-1 + x + 9x^2$ in two different ways. First, use the addition defined for P_2. Second, use T and T^{-1} and make use of the addition defined for $\mathbb{C}^3$. (15 points)

1) In P_2: $(2+3x-5x^2) + (-1+x+9x^2) = 1 + 4x + 4x^2$. Easy.

2) Convert to $\mathbb{C}^3$: $T(2+3x-5x^2) = \begin{bmatrix}-27\\-12\\-20\end{bmatrix}$; $T(-1+x+9x^2) = \begin{bmatrix}64\\28\\45\end{bmatrix}$

Add in $\mathbb{C}^3$: $\begin{bmatrix}-27\\-12\\-20\end{bmatrix} + \begin{bmatrix}64\\28\\45\end{bmatrix} = \begin{bmatrix}37\\16\\25\end{bmatrix}$

Convert to P_2: $T^{-1}\left(\begin{bmatrix}37\\16\\25\end{bmatrix}\right) = 1 + 4x + 4x^2$

5. Suppose that $\dim U = m$, $\dim V = n$, and $T\colon U \to V$ is a linear transformation given by $T(\mathbf{u}) = \mathbf{0}$. (15 points)

(a) Determine the kernel and range of T.

$$K(T) = \{ u \in U \mid T(\underset{\sim}{u}) = \underset{\sim}{0} \} = U$$
$$R(T) = \{ T(\underset{\sim}{u}) \mid u \in U \} = \{\underset{\sim}{0}\}$$

(b) Use your previous answer to compute the nullity and rank of T.

$$n(T) = \dim(K(T)) = \dim(U) = m$$
$$r(T) = \dim(R(T)) = \dim(\{\underset{\sim}{0}\}) = 0$$

(c) Demonstrate how a theorem can provide a quick check on your answer to the previous part.

rank + nullity = dimension of domain (Theorem RPNDD)

$0 + m = m$ ✓

6. Suppose that $T\colon U \to V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ is a basis of U. Prove that if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_m)\}$ is a linearly independent subset of V, then T is injective. (15 points)

Compute $K(T)$. (Trivial?) Grab $\underset{\sim}{u} \in K(T)$.

Write $\underset{\sim}{u} = a_1\underset{\sim}{u}_1 + a_2\underset{\sim}{u}_2 + \cdots + a_m\underset{\sim}{u}_m$ since B spans U

$$\underset{\sim}{0} = T(\underset{\sim}{u}) = T(a_1\underset{\sim}{u}_1 + a_2\underset{\sim}{u}_2 + \cdots + a_m\underset{\sim}{u}_m)$$
$$= a_1T(\underset{\sim}{u}_1) + a_2T(\underset{\sim}{u}_2) + \cdots + a_mT(\underset{\sim}{u}_m)$$

A RLD on the L.I. set C. $\Rightarrow a_1 = a_2 = \cdots = a_m = 0$.

So $\underset{\sim}{u} = 0\underset{\sim}{u}_1 + 0\underset{\sim}{u}_2 + \cdots + 0\underset{\sim}{u}_m = \underset{\sim}{0} + \underset{\sim}{0} + \underset{\sim}{0} + \cdots + \underset{\sim}{0} = \underset{\sim}{0}$

Thus $K(T) = \{\underset{\sim}{0}\}$ & by theorem KILT, T is injective.