Math 290 Name: Dr. Beezer
Exam 6 Spring 2019
Chapter LT

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Verify that the function $T$ is a linear transformation. (15 points)

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \qquad T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}a+b\\b-c\end{bmatrix}$$

2. Answer the following yes/no questions about $T$. Full credit requires a complete and convincing explanation, a simple "yes" or "no" will get no credit. ($P_1$ is the vector space of polynomials with degree at most 1, and $M_{22}$ is teh vector space of $2 \times 2$ matrices.) (15 points)

$$T\colon P_1 \to M_{22}, \qquad T\left(a + bx\right) = \begin{bmatrix}a+2b & 2a+b\\-a+b & a+2b\end{bmatrix}$$

(a) Is $T$ injective?

(b) Is $T$ surjective?

(c) Is $T$ invertible?

3. The linear transformation $T$ below is invertible (you can assume this). Determine a formula for the inverse linear transformation $T^{-1}$. ($P_2$ is the vector space of polynomials with degree at most 2.) (25 points)

$$T\colon P_2 \to \mathbb{C}^3, \qquad T\left(a+bx+cx^2\right) = \begin{bmatrix} a+2b+7c \\ b+3c \\ a+b+5c \end{bmatrix}$$

4. Since the linear transformation $T$ in the previous question is invertible, we know that $P_2$ and $\mathbb{C}^3$ are isomorphic vector spaces. Compute the sum of $2+3x-5x^2$ and $-1+x+9x^2$ in two different ways. First, use the addition defined for $P_2$. Second, use $T$ and $T^{-1}$ and make use of the addition defined for $\mathbb{C}^3$. (15 points)

5. Suppose that $\dim U = m$, $\dim V = n$, and $T\colon U \to V$ is a linear transformation given by $T(\mathbf{u}) = \mathbf{0}$. (15 points)

   (a) Determine the kernel and range of $T$.

   (b) Use your previous answer to compute the nullity and rank of $T$.

   (c) Demonstrate how a theorem can provide a quick check on your answer to the previous part.

6. Suppose that $T\colon U \to V$ is a linear transformation and $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_m\}$ is a basis of $U$. Prove that if $C = \{T(\mathbf{u}_1), T(\mathbf{u}_2), T(\mathbf{u}_3), \ldots, T(\mathbf{u}_m)\}$ is a linearly independent subset of $V$, then $T$ is injective. (15 points)