1. Verify that the function $T$ is a linear transformation. (15 points)

$$T: \mathbb{C}^3 \rightarrow \mathbb{C}^2, \quad T \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a + b \\ b - c \end{bmatrix}$$

2. Answer the following yes/no questions about $T$. Full credit requires a complete and convincing explanation, a simple “yes” or “no” will get no credit. ($P_1$ is the vector space of polynomials with degree at most 1, and $M_{22}$ is the vector space of $2 \times 2$ matrices.) (15 points)

$$T: P_1 \rightarrow M_{22}, \quad T(a + bx) = \begin{bmatrix} a + 2b & 2a + b \\ -a + b & a + 2b \end{bmatrix}$$

(a) Is $T$ injective?

(b) Is $T$ surjective?

(c) Is $T$ invertible?
3. The linear transformation $T$ below is invertible (you can assume this). Determine a formula for the inverse linear transformation $T^{-1}$. ($P_2$ is the vector space of polynomials with degree at most 2.) (25 points)

$$T: P_2 \to \mathbb{C}^3, \quad T(a + bx + cx^2) = \begin{bmatrix} a + 2b + 7c \\ b + 3c \\ a + b + 5c \end{bmatrix}$$

4. Since the linear transformation $T$ in the previous question is invertible, we know that $P_2$ and $\mathbb{C}^3$ are isomorphic vector spaces. Compute the sum of $2 + 3x - 5x^2$ and $-1 + x + 9x^2$ in two different ways. First, use the addition defined for $P_2$. Second, use $T$ and $T^{-1}$ and make use of the addition defined for $\mathbb{C}^3$. (15 points)
5. Suppose that \( \dim U = m \), \( \dim V = n \), and \( T: U \to V \) is a linear transformation given by \( T(u) = 0 \). (15 points)

(a) Determine the kernel and range of \( T \).

(b) Use your previous answer to compute the nullity and rank of \( T \).

(c) Demonstrate how a theorem can provide a quick check on your answer to the previous part.

6. Suppose that \( T: U \to V \) is a linear transformation and \( B = \{u_1, u_2, u_3, \ldots, u_m\} \) is a basis of \( U \). Prove that if \( C = \{T(u_1), T(u_2), T(u_3), \ldots, T(u_m)\} \) is a linearly independent subset of \( V \), then \( T \) is injective. (15 points)