Math 290 Name: Key Dr. Beezer
Exam 5 Spring 2019
Chapters D&E

Show *all* of your work and *explain* your answers fully. There is a total of 90 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. Some problems will permit further, specific Sage functions. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Suppose that B is a 10×10 matrix whose determinant is 7. Compute the following determinants of products of B with elementary matrices. (15 points)

   (a) $\det(E_{3,7}B) = -7$: A row swap changes sign.

   (b) $\det(E_6(4)\,B) = 4 \cdot 7 = 28$: A scalar multiple of a row introduces a factor.

   (c) $\det(E_{5,2}(-9)\,B) = 7$: The third row operation makes no change.

2. Determine if the matrix A is diagonalizable or not. Explain fully and carefully. You may use Sage to obtain a factored characteristic polynomial. (15 points)

$$A = \begin{bmatrix} 32 & -78 & -63 & -5 & 5 & 29 \\ 18 & -32 & -30 & 0 & -6 & 18 \\ -17 & -14 & -1 & -11 & 37 & -16 \\ -24 & 24 & 24 & -2 & 24 & -24 \\ 1 & -50 & -35 & -11 & 35 & 2 \\ -26 & -22 & -7 & -17 & 57 & -21 \end{bmatrix}$$

A.fcp() → (X-3) * (X+2)^2 * (X-4)^3

$\alpha_A(4) = 3$

$(A - 4I_6)$.rref() →

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -4/3 & 2/3 \\ 0 & 1 & 0 & 0 & -1/2 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & -2/3 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

nullity = 6 - 4 = 2

So $\gamma_A(4) = 2$

So the eigenspace is "not large enough."

By Theorem DMFE, the matrix is not diagonalizable

3. Determine the eigenspaces of the matrix A. You may use Sage to obtain a factored characteristic polynomial. (15 points)

$$A = \begin{bmatrix} 7 & 10 & 20 & -10 \\ 5 & -13 & -5 & 40 \\ -5 & 0 & -8 & -10 \\ 0 & -5 & -5 & 12 \end{bmatrix}$$

$A.\text{fcp}() \to (x-2)^2(x+3)^2$

$E_A(2)$? $A - 2I_4 \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ $N(A-2I_4) = \left\langle \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$

$E_A(-3)$ $A - (-3)I_4 \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$ $N(A-(-3)I_4) = \left\langle \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$

4. Find an invertible matrix S and a diagonal matrix D such that $S^{-1}BS = D$. You may use Sage to obtain a factored characteristic polynomial. (15 points)

$$B = \begin{bmatrix} 10 & 4 & 26 \\ 21 & 7 & 54 \\ -7 & -2 & -17 \end{bmatrix}$$

$B.\text{fcp}() = (x-3)(x-1)(x+4)$

So the eigenvalues are $3, 1, -4$ &

$D = \begin{bmatrix} 3 & & \\ & 1 & \\ & & -4 \end{bmatrix}$

Eigenvectors

$B - 3I_3 \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underset{\sim}{x_1} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

$B - 1I_3 \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underset{\sim}{x_2} = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

$B - (-4)I_3 \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underset{\sim}{x_3} = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

So $S = [\underset{\sim}{x_1} | \underset{\sim}{x_2} | \underset{\sim}{x_3}]$

$= \begin{bmatrix} -2 & -2 & -1 \\ -3 & -2 & -3 \\ 1 & 1 & 1 \end{bmatrix}$

Notes

1. Eigenvalues & eigenvectors must be in the same order.
2. No need to check that $S^{-1}BS = D$.

5. Suppose that λ is an eigenvalue of the matrix A. Prove that $\lambda^2 - 6\lambda + 2$ is an eigenvalue of $A^2 - 6A + 2I$, where I is the identity matrix. Do not do this by applying theorems (such as ESMM, EOMP, and EPM), but instead, restrict yourself to the definition of an eignvalue and basic properties of operations with matrices. (15 points)

Let $\underset{\sim}{x}$ be an eigenvector of A for λ. So $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$.
Then

$$
\begin{aligned}
(A^2-6A+2I)\underset{\sim}{x} &= A^2\underset{\sim}{x} - 6A\underset{\sim}{x} + 2I\underset{\sim}{x} \\
&= A(A\underset{\sim}{x}) - 6A\underset{\sim}{x} + 2I\underset{\sim}{x} \\
&= A\lambda\underset{\sim}{x} - 6\lambda\underset{\sim}{x} + 2\underset{\sim}{x} \\
&= \lambda(A\underset{\sim}{x}) - 6\lambda\underset{\sim}{x} + 2\underset{\sim}{x} \\
&= \lambda(\lambda\underset{\sim}{x}) - 6\lambda\underset{\sim}{x} + 2\underset{\sim}{x} \\
&= \lambda^2\underset{\sim}{x} - 6\lambda\underset{\sim}{x} + 2\underset{\sim}{x} \\
&= (\lambda^2 - 6\lambda + 2)\underset{\sim}{x}
\end{aligned}
$$

which establishes that $\lambda^2 - 6\lambda + 2$ is an eigenvalue of A.

6. Prove that a single vector cannot simultaneously be an eigenvector for two different eigenvalues. (15 points)

Suppose $\underset{\sim}{x}$ is an eigenvector of A for λ,
<u>and</u> $\underset{\sim}{x}$ is an eigenvector of A for ρ. Then $\underset{\sim}{x} \neq \underset{\sim}{0}$ &
$(\lambda - \rho)\underset{\sim}{x} = \lambda\underset{\sim}{x} - \rho\underset{\sim}{x} = A\underset{\sim}{x} - A\underset{\sim}{x} = \underset{\sim}{0}$.
Then $\lambda - \rho = 0$, $\lambda = \rho$ which contradicts that λ & ρ should be different. So this cannot happen.