Math 290
Exam 4
Chapter VS

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Determine if the set of matrices, T, from the vector space M_{22}, is a linearly independent set. (15 points)

$$T = \left\{ \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \right\}$$

$$\alpha_1 \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 + 2\alpha_2 + 4\alpha_3 & -2\alpha_1 - \alpha_2 + \alpha_3 \\ \alpha_1 \quad -2\alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

leads to a homogeneous system with coefficient matrix,

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\alpha_1 = 2$, $\alpha_2 = -3$, $\alpha_3 = 1$ is a non-trivial solution and will create a non-trivial RLD, so T is linearly dependent.

2. For the matrix A, compute the dimension of the column space and the dimension of the row space. Your answer should illustrate an important theorem from this chapter. State the relevant theorem (in other words, write out the theorem, do not just quote the acronym.) (15 points)

$$A = \begin{bmatrix} 1 & 0 & 2 & -6 & -4 & -3 \\ -1 & 1 & 0 & -2 & 5 & 0 \\ 0 & 0 & 1 & -2 & -1 & -1 \\ 1 & -1 & 3 & -4 & -8 & -3 \end{bmatrix}$$

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -2 & -2 & -1 \\ 0 & 1 & 0 & -4 & 3 & -1 \\ 0 & 0 & 1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3 rows as column vectors are a basis of row space $\Rightarrow \dim(\mathcal{R}(A)) = 3$

$$A^t \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 rows as column vectors are a basis of column space $\Rightarrow \dim(\mathcal{C}(A)) = 3$

Theorem RMRT The rank of a matrix is equal to the rank of the transpose of the matrix.

So the dimensions above must be equal.

3. Y is a subspace of P_2, the vector space of polynomials with degree at most 2. (You may assume this much.) Answer the following ques~~s~~tions with *complete* justifications. (40 points)

$Y = \{a + bx + cx^2 \,|\, a - b + 2c = 0\}$

(a) Prove that the dimension of Y is 2, $\dim(Y) = 2$.

$Y = \{(b-2c) + bx + cx^2 \,|\, b, c \in \mathbb{C}\} = \{b(1+x) + c(-2+x^2) \,|\, b, c \in \mathbb{C}\}$

$= \langle\{1+x, -2+x^2\}\rangle$. Is this spanning set also linearly independent?

$\alpha_1(1+x) + \alpha_2(-2+x^2) = 0 + 0x + 0x^2 \rightarrow \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \alpha_1 = \alpha_2 = 0$ yes.

coefficient matrix of a homogeneous system.

A basis w/ 2 vectors $\Rightarrow \dim(Y) = 2$.

(b) Is $\{4 + 2x - x^2, 5 + 3x - x^2\}$ a basis of Y?

Homogeneous system:

$\begin{bmatrix} 4 & 5 \\ 2 & 3 \\ -1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow$ set is linearly independent. $\Rightarrow$ By Theorem G, spans Y. Thus a basis.

(c) Is $\{1 + 3x + x^2, 2 + 6x + 2x^2\}$ a basis of Y?

Homogenous system

$\begin{bmatrix} 1 & 2 \\ 3 & 6 \\ 1 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$ set is linearly dependent $\Rightarrow$ Does not meet definition of a basis.

(d) Is $\{-5 - x + 2x^2\}$ a basis of Y?

Too small. $\dim(Y) = 2$, size of set $= 1$, so Theorem G says the set does not span Y, so is not a basis of Y.

(e) Is $\{-2 + 2x + 2x^2, 3 + x - 2x^2\}$ a basis of Y?

$3 + x - 2x^2 \notin Y$ $\quad 3 - (1) + 2(-2) = -2 \neq 0$.

So can't be a basis.

4. The set W is a subset of the vector space of column vectors, $\mathbb{C}^3$. Give a careful proof that W is a subspace of $\mathbb{C}^3$. (15 points)

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, 2a - 3c = 0 \right\}$$

Apply the three-part test of Theorem TSS.

① $\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$ since $2(0) - 3(0) = 0$. So $W \neq \emptyset$.

② Suppose $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in W$, $\underset{\sim}{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in W$. Then $\underset{\sim}{x} + \underset{\sim}{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$

Know $2x_1 - 3x_3 = 0$ & $2y_1 - 3y_3 = 0$.

Consider $2(x_1 + y_1) - 3(x_3 + y_3) = 2x_1 + 2y_1 - 3x_3 - 3y_3$

$= (2x_1 - 3x_3) + (2y_1 - 3y_3) = 0 + 0 = 0$

So $\underset{\sim}{x} + \underset{\sim}{y} \in W$.

③ Suppose $\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in W$, $\alpha \in \mathbb{C}$. Then $\alpha \underset{\sim}{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}$.

And know $2x_1 - 3x_3 = 0$. Examine

$2(\alpha x_1) - 3(\alpha x_3) = \alpha(2x_1 - 3x_3) = \alpha \cdot 0 = 0$.

So $\alpha \underset{\sim}{x} \in W$.

5. Suppose that V is a vector space and that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ is a subset of V. Prove that the span of S, $\langle V \rangle$, is a subspace of V. (15 points)

This is Theorem SSS in the book.

Use Theorem TSS as the main tool, as in the proof given there.