1. Solve the following system of linear equations and express the solutions as a set of column vectors. (15 points)

\[-3x_1 - 3x_2 - 2x_3 + 2x_4 = 2\]
\[2x_1 + 3x_2 + x_3 - 2x_4 = 8\]
\[-4x_1 - 4x_2 - 3x_3 + x_4 = -7\]
\[-3x_1 - 5x_2 - 2x_3 + x_4 = 3\]

Augmented matrix of the system:

\[
\begin{bmatrix}
-3 & -3 & -2 & 2 & 2 \\
2 & 3 & 1 & -2 & 6 \\
-4 & -4 & -3 & 1 & -7 \\
-3 & -5 & -2 & 1 & 3 \\
\end{bmatrix}
\]

Equivalent system:
\[
\begin{align*}
x_1 &= 156 \\
x_2 &= -20 \\
x_3 &= -166 \\
x_4 &= 39
\end{align*}
\]

Unique solution

Answer:
\[
\begin{bmatrix}
156 \\
-20 \\
-166 \\
39
\end{bmatrix}
\]

2. Solve the following system of linear equations and express the solutions as a set of column vectors. (15 points)

\[-x_1 + 7x_2 - 8x_3 + 3x_4 = 3\]
\[-x_1 + 6x_2 - 7x_3 + 3x_4 = 1\]
\[-5x_2 + 5x_3 = 2\]
\[x_1 - 7x_2 + 8x_3 - 3x_4 = 1\]

Augmented matrix of the system:

\[
\begin{bmatrix}
-1 & 7 & -8 & 3 & 3 \\
-1 & 6 & -7 & 3 & 1 \\
0 & -5 & 5 & 0 & 2 \\
1 & -7 & 8 & -3 & 1
\end{bmatrix}
\]

Last column is a pivot column. By Theorem RCLS there are no solutions.
3. Find the null space of the matrix $B$, $\mathcal{N}(B)$. (15 points)

\[ B = \begin{bmatrix} -5 & 3 & -7 & 4 \\ -2 & 1 & -3 & 2 \\ 1 & 1 & 3 & -4 \end{bmatrix} \quad \xrightarrow{RREF} \quad \begin{bmatrix} 0 & 0 & 2 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

$\mathsf{LS}(B, \mathbf{0})$ is equivalent to $x_1 - 2x_3 - 2x_4 = 0$

$x_4 = x_3 - 2x_4$

is equivalent to $x_1 = -2x_3 + 2x_4$

\[ x_2 = -x_3 + 2x_4 \]

Answer:

\[
\mathcal{N}(B) = \left\{ \begin{bmatrix} -2x_3 + 2x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} \right\}
\]

4. Without using Sage, find a matrix $B$ in reduced row-echelon form which is row-equivalent to $A$. It is especially important to show all of your work, so it is clear you have not used Sage. (15 points)

\[ A = \begin{bmatrix} 1 & 1 & -3 \\ 3 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \]

\[ -3R_1 + R_2 \Rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & -2 & 10 \\ 0 & 1 & -5 \end{bmatrix} \]

\[ -\frac{1}{2}R_2 \Rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & -1 & 5 \\ 0 & 1 & -5 \end{bmatrix} \]

\[ -R_2 + R_1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \]

Answer:
5. Each of the following statements is false or incorrect in some way. Identify and explain the problem, citing a theorem or definition if relevant. Can you “fix” the statement with a small change? If so, then do so. (30 points)

(a) $A$ is a nonsingular matrix and $\mathbf{LS}(A, 0)$ has infinitely many solutions.

The definition of nonsingular says this system has just one solution. Fix: nonsingular $\rightarrow$ singular or infinitely many $\rightarrow$ no

(b) $\mathbf{LS}(B, 0)$ is a consistent linear system of 9 equations with 12 variables and a unique solution.

Theorem CMVEII says the system has infinitely many solutions

Fix: unique solution $\rightarrow$ infinitely many solutions

(c) $\mathbf{LS}(C, 0)$ is a consistent linear system of 12 equations with 9 variables and no solutions.

"Consistent" and "no solutions" are contradictory.

$\mathbf{LS}(C, 0)$ is always consistent, Theorem HSC.

(d) $D$ is a $5 \times 5$ nonsingular matrix, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{LS}(D, \mathbf{b})$ has no solutions.

Theorem NMUS says $\mathbf{LS}(\mathbf{D}, \mathbf{b})$ has a unique solution.

Fix: no solutions $\rightarrow$ unique solutions.

(e) $E$ is a $15 \times 3$ matrix with $\mathcal{N}(E) = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}$.

A null space always contains the zero vector, here $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(f) $F$ is a $4 \times 12$ matrix with $\mathcal{N}(F) = \{\}$.

As above in (e) the null space cannot ever be empty.