Math 290
Exam 1
Chapter SLE

Name: Key

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Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices, except in the question that asks you to row-reduce without Sage. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Solve the following system of linear equations and express the solutions as a set of column vectors. (15 points)

$$\begin{aligned} -3x_1 - 3x_2 - 2x_3 + 2x_4 &= 2 \\ 2x_1 + 3x_2 + x_3 - 2x_4 &= 8 \\ -4x_1 - 4x_2 - 3x_3 + x_4 &= -7 \\ -3x_1 - 5x_2 - 2x_3 + x_4 &= 3 \end{aligned}$$

Augmented matrix of the system:

$$\left[\begin{array}{cccc|c} -3 & -3 & -2 & 2 & 2 \\ 2 & 3 & 1 & -2 & 8 \\ -4 & -4 & -3 & 1 & -7 \\ -3 & -5 & -2 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 156 \\ 0 & 1 & 0 & 0 & -20 \\ 0 & 0 & 1 & 0 & -166 \\ 0 & 0 & 0 & 1 & 39 \end{array}\right]$$

Equivalent system:

$x_1 = 156$
$x_2 = -20$
$x_3 = -166$
$x_4 = 39$

Unique solution

Answer: $\left\{\begin{bmatrix} 156 \\ -20 \\ -166 \\ 39 \end{bmatrix}\right\}$

2. Solve the following system of linear equations and express the solutions as a set of column vectors. (15 points)

$$\begin{aligned} -x_1 + 7x_2 - 8x_3 + 3x_4 &= 3 \\ -x_1 + 6x_2 - 7x_3 + 3x_4 &= 1 \\ -5x_2 + 5x_3 \quad &= 2 \\ x_1 - 7x_2 + 8x_3 - 3x_4 &= 1 \end{aligned}$$

Augmented matrix of the system:

$$\left[\begin{array}{cccc|c} -1 & 7 & -8 & 3 & 3 \\ -1 & 6 & -7 & 3 & 1 \\ 0 & -5 & 5 & 0 & 2 \\ 1 & -7 & 8 & -3 & 1 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -3 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Last column is a pivot column. By Theorem RCLS there are no solutions.

Answer: $\{\ \}$ or $\emptyset$

3. Find the null space of the matrix B, $\mathcal{N}(B)$. (15 points)

$$B = \begin{bmatrix} -5 & 3 & -7 & 4 \\ -2 & 1 & -3 & 2 \\ 1 & 1 & 3 & -4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$LS(B, 0)$ is equivalent to $\begin{aligned} x_1 + 2x_3 - 2x_4 &= 0 \\ x_2 + x_3 - 2x_4 &= 0 \end{aligned}$,

is equivalent to $\begin{aligned} x_1 &= -2x_3 + 2x_4 \\ x_2 &= -x_3 + 2x_4 \end{aligned}$

Answer:

$$\mathcal{N}(B) = \left\{ \begin{bmatrix} -2x_3 + 2x_4 \\ -x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle| \; x_3, x_4 \in \mathbb{C} \right\}$$

4. Without using Sage, find a matrix B in reduced row-echelon form which is row-equivalent to A. It is especially important to show all of your work, so it is clear you have not used Sage. (15 points)

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 3 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \xrightarrow[-2R_1 + R_3]{-3R_1 + R_2} \begin{bmatrix} 1 & 1 & -3 \\ 0 & -2 & 10 \\ 0 & -1 & 5 \end{bmatrix} \xrightarrow[-1 R_3]{-1/2 R_2} \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \end{bmatrix}$$

$$\xrightarrow[-1 R_2 + R_3]{-1 R_2 + R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

Answer:

5. Each of the following statements is false or incorrect in some way. Identify and explain the problem, citing a theorem or definition if relevant. Can you "fix" the statement with a small change? If so, then do so. (30 points)

(a) A is a nonsingular matrix and $\mathcal{LS}(A, \mathbf{0})$ has infinitely many solutions.

The definition of nonsingular says this system has just one solution. Fixes: nonsingular → singular
or infinitely many → no

(b) $\mathcal{LS}(B, \mathbf{0})$ is a consistent linear system of 9 equations with 12 variables and a unique solution.

Theorem CMVEI says the system has infinitely many solutions
Fix: unique solution → infinitely many solutions

(c) $\mathcal{LS}(C, \mathbf{0})$ is a consistent linear system of 12 equations with 9 variables and no solutions.

"Consistent" and "no solutions" are contradictory.
$LS(C, \underset{\sim}{0})$ is always consistent, Theorem HSC.

(d) D is a 5×5 nonsingular matrix, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathcal{LS}(D, \mathbf{b})$ has no solutions.

Theorem NMUS says $LS(D, \underset{\sim}{b})$ has a unique solution.
Fix: no solutions → unique solutions.

(e) E is a 15×3 matrix with $\mathcal{N}(E) = \left\{ \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix} \right\}$.

A null space always contains the zero vector, here $\underset{\sim}{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(f) F is a 4×12 matrix with $\mathcal{N}(F) = \{\}$.

As above in (e) the null space cannot ever be empty.