Math 290 Name: Key Dr. Beezer
Exam 5 Fall 2019
Chapters D & E

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. Allowable uses of Sage as justification for your answers is stated in each such problem. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Compute, by hand, the determinant of the matrix A. Include enough detail to show that you have not used Sage. (15 points)

$$A = \begin{bmatrix} 4 & -5 & -2 \\ 2 & 2 & -2 \\ -1 & 2 & 0 \end{bmatrix}$$

Expand about row 3 or column 3 to exploit zero entry

Row 3: $(-1)(+1)\begin{vmatrix} -5 & -2 \\ 2 & -2 \end{vmatrix} + 2(-1)\begin{vmatrix} 4 & -2 \\ 2 & -2 \end{vmatrix} + 0(\quad)$

$= -(10-(-4)) + (-2)(-8-(-4)) = -14 + 8 = -6$

2. Is there a diagonal matrix, D, similar to C? Why or why not? If there is such a matrix D, provide it. You may use Sage commands to obtain information about eigenvalues, eigenvectors, and eigenspaces, but be certain to provide an explanation of how your Sage results *help* you answer the question. (15 points)

$$A = \begin{bmatrix} 53 & 8 & 32 & 8 & 136 & -112 \\ 28 & 33 & 24 & 52 & 28 & 32 \\ -63 & -36 & -43 & -55 & -111 & 60 \\ -21 & -12 & -16 & -16 & -45 & 12 \\ 0 & 4 & 0 & 8 & -11 & 8 \\ 7 & 0 & 4 & -1 & 19 & -19 \end{bmatrix}$$

This is a Sage-heavy answer, there are other approaches.

Compute A.eigenmatrix_right()

Results are a diagonal matrix D and a matrix S. Then $AS = SD$. Row-reduce S and get I_6, so S is non-singular and we have diagonalization: $S^{-1}AS = D$.

Or, do something similar & note that Sage's S has no zero columns and thus Theorem DMFE applies

$$D = \begin{bmatrix} 4 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & -3 & & \\ & & & & -3 & \\ 0 & & & & & -3 \end{bmatrix}$$

3. Conside the matrix F below. You can use Sage on this question to manipulate and row-reduce matrices, but the only new commands you can use as justification will compute and factor the characteristic polynomial. (40 points)

$$F = \begin{bmatrix} -2 & -6 & 5 & 4 \\ -1 & 3 & -5 & -4 \\ 3 & 7 & -13 & -13 \\ -3 & -7 & 10 & 10 \end{bmatrix}$$

(a) Find the eigenvalues of F and the algebraic multiplicities.

$F.\mathrm{fcp}() \rightarrow (x-2)^2 (x+3)^2$

$\alpha(2)=2, \quad \alpha(-3)=2$

(b) Find the eigenspaces of F and the geometric multiplicities.

$$(F-2I).\mathrm{rref}() = \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{E}(2) = \left\langle \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ -1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$\gamma(2)=1$

$$(F-(-3)I).\mathrm{rref}() = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{E}(-3) = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$\gamma(-3)=2$

(c) Is F diagonalizable? Why or why not?

No. $\gamma(2)=1<2=\alpha(2)$, so Theorem DMFE says F is not diagonalizable

4. Suppose A is a square matrix whose determinant is 27. Form a matrix B, of the same size, by starting with A and relocate the first row to become the last row. (15 points)

(a) What is the determinant of B, if A is an 800×800 matrix? Explain your reasoning.

Create B with 799 row swaps.

$\det(B) = (-1)^{799} \det(A) = -27$

(b) What is the determinant of B, if A is an 801×801 matrix? Explain your reasoning.

Create B with 800 row swaps.

$\det B = (-1)^{800} \det(A) = 27$

(c) Formulate a careful and self-contained theorem that you now belive to be true, based the two previous parts. You do not need to supply a proof.

Suppose A is an $n \times n$ matrix, and B is the matrix created from A by relocating the first row to be the last row. If n is odd, then $\det(A) = \det(B)$, while if n is even, then $\det(A) = -\det(B)$

5. Suppose that A and B are similar matrices. Prove that $A^3 + 6A^2 + 3A$ and $B^3 + 6B^2 + 3B$ are similar matrices. (15 points)

Hypothesis says there is a matrix S with $A = S^{-1}BS$.

Then

$$
\begin{aligned}
A^3 + 6A^2 + 3A &= (S^{-1}BS)^3 + 6(S^{-1}BS)^2 + 3(S^{-1}BS) \\
&= S^{-1}BS\,S^{-1}BS\,S^{-1}BS + 6\,S^{-1}BS\,S^{-1}BS + 3S^{-1}BS \\
&= S^{-1}BBBS + 6S^{-1}BBS + 3S^{-1}BS \\
&= S^{-1}B^3S + S^{-1}(6B^2)S + S^{-1}(3B)S \\
&= S^{-1}(B^3 + 6B^2 + 3B)S
\end{aligned}
$$

establishing the desired similarity.