Math 290 Name: Key Dr. Beezer
Exam 3 Fall 2019
Chapter M

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. You may use Sage to row-reduce matrices, and note that one problem requires you to not use Sage at all. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Find the solutions of the following system by employing an inverse of the coefficient matrix. **No credit** will be given for solutions obtained by another method. (15 points)

$$\begin{aligned} -x_1 + 5x_2 - 5x_3 &= 3 \\ -x_1 + 4x_2 - 4x_3 &= 2 \\ 3x_1 - 4x_2 + 5x_3 &= 3 \end{aligned}$$

Via theorem SLEMM, this is the system

$A\underset{\sim}{x} = \underset{\sim}{b}$, $A = \begin{bmatrix} -1 & 5 & -5 \\ -1 & 4 & -4 \\ 3 & -4 & 5 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$

A solution (~~the~~ solution) is $A^{-1}\underset{\sim}{b}$.

Theorem CINM: $[A \mid I_3] \xrightarrow{RREF} \left[I_3 \;\middle|\; \underbrace{\begin{matrix} 4 & -5 & 0 \\ -7 & 10 & 1 \\ -8 & 11 & 1 \end{matrix}}_{A^{-1}} \right]$

$A^{-1}\underset{\sim}{b} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ (which can be easily checked)

A matrix-vector product

2. Given the matrices A and B below, compute $[AB]_{3,5}$. (15 points) (Directions said, "No Sage.")

$$A = \begin{bmatrix} -4 & 1 & -1 & 3 \\ 6 & -5 & 2 & 9 \\ 1 & 4 & 4 & -4 \\ -6 & -9 & -2 & 5 \\ 9 & -8 & 2 & 7 \\ -2 & 9 & 6 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} -7 & -5 & 4 & 1 & -2 & 2 & -4 \\ -1 & 3 & 4 & -2 & -4 & -2 & 3 \\ 0 & 3 & -2 & -8 & 8 & 0 & -2 \\ -3 & -5 & -6 & 2 & 6 & -1 & -4 \end{bmatrix}$$

Theorem EMP says we use row 3 of A & column 5 of B to sum 4 products.

$$\begin{aligned} [AB]_{3,5} &= 1(-2) + 4(-4) + 4(8) + (-4)(6) \\ &= -2 + (-16) + 32 + (-24) \\ &= -10 \end{aligned}$$

3. In each part of this question, find a set S of column vectors so that the span of S equals the column space of the matrix A (that is, $\mathcal{C}(A) = \langle C \rangle$) and S satisfies the additional requirements given in each part. Provide clear explanations on why your answer meets all the requirements. (40 points)

$$A = \begin{bmatrix} -5 & -10 & 15 & -17 & -46 & -15 \\ 7 & 14 & -21 & 24 & 65 & 21 \\ 3 & 6 & -9 & 10 & 27 & 9 \\ 2 & 4 & -6 & 7 & 19 & 6 \end{bmatrix}$$

(a) S is a consequence of the **definition** of a column space and requires no computation to create.

The definition of the column space is the span of the columns.

$$S = \left\{ \begin{bmatrix} -5 \\ 7 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -10 \\ 14 \\ 6 \\ 4 \end{bmatrix}, \ldots, \begin{bmatrix} -15 \\ 21 \\ 9 \\ 6 \end{bmatrix} \right\}$$

(b) S is linearly independent and is a subset of the set of columns of A.

$$A \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & -3 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$D = \{1, 4\}$ & Theorem BCS says we can use these two columns of the original set of columns to meet the requirements.

$$S = \left\{ \begin{bmatrix} -5 \\ 7 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -17 \\ 24 \\ 10 \\ 7 \end{bmatrix} \right\}$$

(c) S is linearly independent and is computed using results about row spaces.

Theorem CSRST $\Rightarrow \mathcal{C}(A) = \mathcal{R}(A^t)$ & $A^t \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Theorem BRS says we meet the requirements by "keeping" the non-zero rows as column vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

(d) S is linearly independent and is computed using the matrix L from the extended echelon form of A.

$$[A \mid I_4] \xrightarrow{\text{RREF}} \left[\begin{array}{cccccc|cccc} 1 & 2 & 3 & 0 & -1 & 3 & 0 & 0 & 7 & -10 \\ 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 \end{array}\right] \Rightarrow L = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -2 \end{bmatrix}$$

Theorem FS $\Rightarrow \mathcal{C}(A) = \mathcal{N}(L)$

Theorem BNS $\Rightarrow$ $$S = \left\{ \begin{bmatrix} \\ \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \\ \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4. Suppose that A and B are both $m \times n$ matrices. Prove that $A + B = B + A$. (15 points)

For $1 \le i \le m$, $1 \le j \le n$,

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij} \qquad \text{Definition MA}$$
$$= [B]_{ij} + [A]_{ij} \qquad \text{Scalar commutativity}$$
$$= [B+A]_{ij} \qquad \text{Definition MA}$$

So, by Definition ME, $A+B = B+A$.

5. Suppose that A is an $m \times n$ matrix. Form $M = [A|I_m]$, and then there is a row-equivalent matrix in reduced row-echelon form, $N = [B|J]$ (that is, N is "extended echelon form"). Starting with just this definition of J, give a careful and complete proof that J is nonsingular. (15 points)

1. the sequence of row operations which convert M to N, will convert I_m to J.

2. If we reverse this sequence (modifying the row operations as we go) then we can form a sequence of row operations which converts J to I_m.

3. By Theorem NMRRI, J is nonsingular

The first paragraph of the proof of Theorem PEEF has a proof that is more careful than part 2 above.