Lie, Noether, and Lagrange

symmetries, and their relation to conserved quantities

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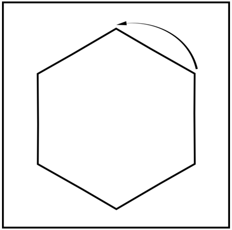
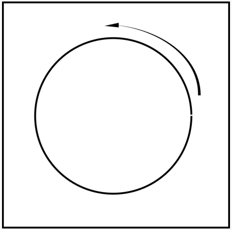
Introduction: Discrete v. Continuous





- Permutation groups are the language of discrete symmetries.
  - The symmetries of a hexagon in the plane are represented by $\mathbb{Z}_6$.
- Lie groups allow us to talk about continuous symmetries.
  - The symmetries of a circle, on the other hand, cannot be represented by a finite group.
  - We need to develop Lie groups in order to describe them.

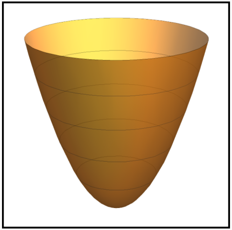
Differentiable Manifolds

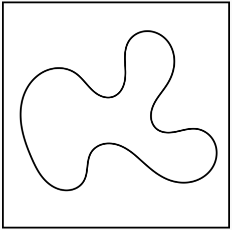
Differentiable Manifolds

- Differentiable Manifolds are smooth surfaces of arbitrary dimension.
- They can live in $\mathbb{C}^n$ or $\mathbb{R}^n$ (but for simplicity, I will use $\mathbb{R}^n$).
- In the vicinity of any point, the manifold approximates Cartesian space.
- There is a tangent space corresponding to each point.

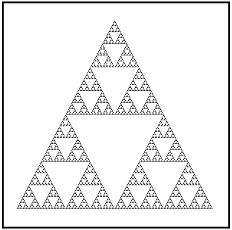
Examples and Non-Examples

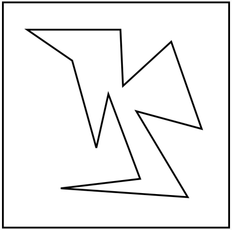
Examples





Non-Examples





coordinates

- It is useful to know where on a manifold we are.
- If we write a manifold X as

$$X = \{x(q_1, q_2, \dots, q_n)\} = \{x(q_i)\},$$

then we call q_i the generalized coordinate.

- If you need n generalized coordinates to define a manifold, then it is an n dimensional manifold.

Lie Groups

Lie Groups

- A Lie group is a group over a differentiable manifold G.
- The binary operation of the group is defined by the differentiable function

$$\mu : G \times G \to G \qquad \mu(p_1, p_2) = p_3.$$

- The operation μ must be associative and have an identity.
- The inverse of a point is defined by the differentiable function

$$\iota : G \to G \qquad \iota(p) = p^{-1}$$

Example: Circle (Part 1)

- Points in a circle are points of the form:

$$p(\theta) = r_0 \cdot \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}, \quad \theta \in \mathbb{R}$$

- We define multiplication as

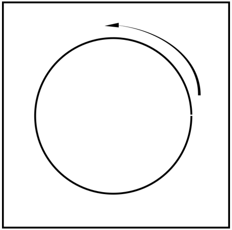
$$\mu(p(\theta), p(\phi)) = p(\theta + \phi)$$

.

- The inverse of a point is

$$\iota(p(\theta)) = p(-\theta)$$

.



Example: Circle (Part 2)

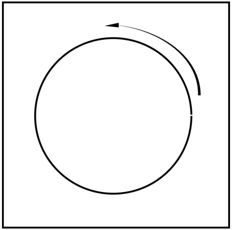
- Both of these functions are everywhere differentiable:

$$\frac{\partial}{\partial\theta}\mu(p(\theta), p(\phi)) = \frac{\partial}{\partial\theta}p(\theta+\phi) = r_0 \cdot \begin{pmatrix} -\sin(\theta+\phi) \\ \cos(\theta+\phi) \end{pmatrix},$$

with differentiation with respect to ϕ yielding similar results.

- For inverses,

$$\frac{d}{d\theta}\iota(p(\theta)) = \frac{d}{d\theta}r_0 \cdot \begin{pmatrix} \cos(-\theta) \\ \sin(-\theta) \end{pmatrix} = r_0 \cdot \begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \end{pmatrix}$$



Tangent Algebras

Tangent Algebras

- Because Lie Groups are groups on differentiable manifolds, every element of the Lie group has a tangent space.
- We can turn each tangent space into a Lie group, with the point generating the tangent space as the identity.
- This new Lie group is called the tangent algebra of the original Lie group.
- There is a homomorphism between a Lie group and its tangent group for points local to the generating point.

Again, but with math

- Formally, if the full Lie group depends on parameters ϵ_i, then the tangent algebra to the point p in G is the set

$$\{p + \sum_i \frac{\partial G}{\partial \epsilon_i}\Big|_p \varepsilon_i \mid \varepsilon_i \in \mathbb{R}\}$$

- This is identical to doing a Taylor expansion of G and throwing out all of the higher power terms.
- For compactness, we write

$$\frac{\partial G}{\partial \epsilon_i}\Big|_p = \zeta_i$$

More Circles

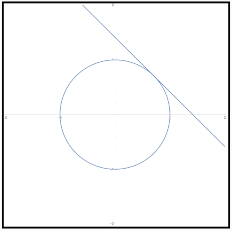
- For a circle, the line tangent to a point $p(\theta)$ is the set:

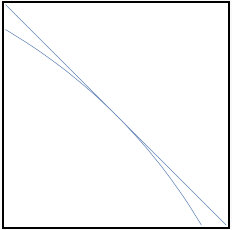
$$\{r_0 \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} + \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} t \mid t \in \mathbb{R}\}.$$

- We can define multiplication of points in the tangent line to be

$$\mu'(p'(s), p'(t)) = p'(s+t).$$

- For small t, $p'(t) \approx p(\theta + t)$.





Lie Group Actions

Lie Group Actions

- Lie group actions are ways of talking about the symmetries of manifolds that are not Lie groups.
- If there is a manifold X, then the action of a Lie group G on X is a differentiable function

$$\alpha : G \times X \to X \qquad (g, x) \to \alpha(g)x$$

- Each element of the Lie group is a symmetry of the manifold X.
- If $x(q_i)$ is a point in the manifold X, then

$$\alpha(g)x(q_i) = x(Q_{g,i}(q_j))$$

Local Actions

- Just as Lie groups have tangent groups, we can define a local action of a Lie group on a manifold.
- Recall, the tangent algebra is the set

$$\{p + \sum_i \zeta_i \varepsilon_i \mid \varepsilon_i \in \mathbb{R}\}.$$

- The action is

$$\alpha(g)x \approx \alpha(\sum_i \zeta_i \varepsilon_i)x$$

for g close to the identity of the Lie group.

Example: Symmetries of a Paraboloid

- Our Lie group is the group on a circle we have already defined.
- Our Lie group X is the paraboloid

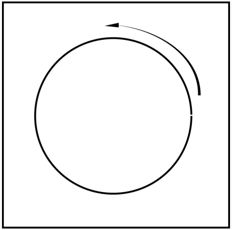
$$X = \{z = x^2 + y^2 \mid x, y \in \mathbb{R}\}.$$

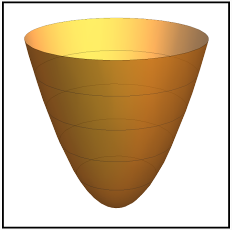
- We can define the action

$$\alpha(p(\theta))(\begin{pmatrix} x \\ y \\ x^2+y^2 \end{pmatrix}) = \begin{pmatrix} x\cos(\theta) + y\sin(\theta) \\ y\cos(\theta) - x\sin(\theta) \\ x^2+y^2 \end{pmatrix}$$

- The local action is

$$\alpha(p(\varepsilon))(\begin{pmatrix} x \\ y \\ x^2+y^2 \end{pmatrix}) = \begin{pmatrix} x + y\,\varepsilon \\ y - x\,\varepsilon \\ x^2+y^2 \end{pmatrix}.$$





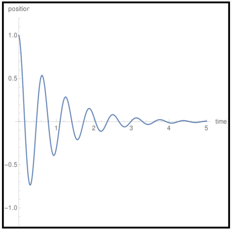
A Side-note: Representation

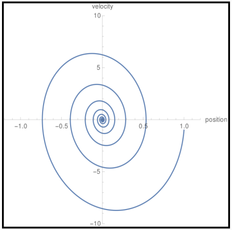
- Every finite group is isomorphic to a subgroup of S_n.
- Every Lie group is isomorphic to a subgroup of $GL(n)$, the group of n-dimensional invertible matrices.
- For example, the Lie group on a circle is isomorphic to

$$\{\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \mid \theta \in \mathbb{R}\}.$$

Lagrangian Mechanics

Phase Space





- Phase space is set of all possible states a physical system can be in.
- Half of the coordinates denote the position of particles while the other half denote the velocities.
- We denote position in phase space as a point $(q_i, \dot{q}_i)$.

The Lagrangian

- ➢ The Lagrangian ($\mathcal{L}(q_i, \dot{q}_i, t)$) is a function of position in phase space and in time.
- ➢ The Lagrangian is the difference between the kinetic and potential energies.
- ➢ Given a Lagrangian, we can use the Euler-Lagrange equations to find the evolution of a system in time.
- ➢ The Lagrangian is a differentiable manifold.

Noether's Theorem

The Theorem

- Let G be a Lie group that acts on the Lagrangian $\mathcal{L}(q_i, \dot{q}_i, t)$.
- If the action of the Lie Group on the Lagrangian is

$$\alpha(g)\mathcal{L}(q_i, \dot{q}_i, t) = \mathcal{L}(Q_{g,i}(q_j, \dot{q}_j, t), Q'_{g,i}(q_j, \dot{q}_j, t), T_g(q_j, \dot{q}_j, t)),$$

with local symmetry

$$\mathcal{L}(q_i + \zeta_i \epsilon, \dot{q}_i + \zeta'_i \epsilon, t + \tau\epsilon)$$

then the quantity $\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\zeta_i - \dot{q}_i \tau) + \mathcal{L}\tau$ is conserved in time.

Conservation of Energy

- If the Lagrangian is not a function of time, then it is invariant under a shift in time.
- Thus $\zeta_i = 0$ and $\tau = -1$.
- By Noether's theorem,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i}(\zeta_i - \dot{q}_i \tau) + \mathcal{L}\tau = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i - \mathcal{L}$$

is conserved.

- This quantity is the energy.

End Notes

This presentation is set in LaTeX, and the theme is **metropolis** by Matthias Vogelgesang.

I heavilly used the books:

- Onishchik and Vinberg's *Lie Groups and Albegraic Groups*
- Neuenschwander's *Emmy Noethers Wonderful Theorem*
- Jones' *Groups, Representations, and Physics*

Questions?