1. The function $T$ below is a linear transformation (you may assume this). (20 points)

$$T : \mathbb{C}^2 \rightarrow \mathbb{C}^3, \quad T \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a - b \\ a + b \\ a \end{bmatrix}$$

(a) For $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \in \mathbb{C}^2$, compute $T(v)$.

(b) Compute the vector representation of $v$, $\rho_B(v)$, relative to the basis $B = \left\{ \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \right\}$.

(c) Compute the matrix representation of $T$, $M_{B,C}^T$, relative to the bases $B$ (above) and $C$ (below).

$$C = \left\{ \begin{bmatrix} 1 \\ -4 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} \right\}$$

(d) Duplicate the computation of $T(v)$ from part (a), but illustrate a non-trivial use of the Fundamental Theorem of Matrix Representation with the representations from parts (b) and (c). You should, of course, get the same answer as part (a), but that is not the real purpose of the question.
2. The function $S$ below is a linear transformation (you may assume this). ($P_2$ is the vector space of polynomials with degree 2 or less, and $M_{1,2}$ is the vector space of $1 \times 2$ matrices.) (40 points)

$$S: P_2 \rightarrow M_{1,2}, \quad S(a + bx + cx^2) = \begin{bmatrix} 2a + b + 2c \quad -a + 3b + c \end{bmatrix}$$

(a) Choose a “nice” basis $B$ for $P_2$, a “nice” basis $C$ for $M_{1,2}$, and then compute the matrix representation of $S$ relative to $B$ and $C$, $M_{B,C}^S$.

(b) Using the bases $D$ and $E$ below, of $P_2$ and $M_{1,2}$ respectively, use the definition of a matrix representation to compute the matrix representation of $S$ relative to $D$ and $E$, $M_{D,E}^S$.

$$D = \{1 + x + 4x^2, 1 - x - 3x^2, -x - 4x^2\} \quad E = \{\begin{bmatrix} -3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -3 \end{bmatrix}\}$$

(c) Use change of basis matrices and your representation in part (a) to re-compute the representation in part (b). Show the relevant product of matrices and compute the necessary products to duplicate the earlier result.
3. The function $S$ below is an invertible linear transformation (you may assume this). Use a matrix representation to compute the inverse linear transformation, $S^{-1}$. No credit will be given for a method that does not make significant use of the matrix representation. ($P_1$ is the vector space of polynomials with degree 1 or less, and $M_{1,2}$ is the vector space of $1 \times 2$ matrices.) (20 points)

$S: P_1 \rightarrow M_{1,2} \quad S (a + bx) = \begin{bmatrix} 2a + 5b & a + 3b \end{bmatrix}$

4. Find a basis for $P_2$ that will yield a matrix representation of $T$ that is a diagonal matrix, along with a complete explanation. ($P_2$ is the vector space of polynomials with degree 2 or less.) (20 points)

$T: P_2 \rightarrow P_2, \quad T (a + bx + cx^2) = (5a + 6b + 12c) + (-9a - 10b - 18c)x + (3a + 3b + 5c)x^2$