

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points. Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. The function T below is a linear transformation (you may assume this). Use a well-defined procedure to compute a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$. No credit will be given for an answer that does not demonstrate the use of a theorem or definition that provides a “recipe” for determining this matrix. (10 points)

$$T: \mathbb{C}^3 \rightarrow \mathbb{C}^2, \quad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2a - 3b + 4c \\ a - b + 8c \end{bmatrix}$$

2. The function S below is an invertible linear transformation (you may assume this). Use a well-defined procedure to compute the inverse linear transformation, S^{-1} . (P_1 is the vector space of polynomials with degree 1 or less, and $M_{1,2}$ is the vector space of 1×2 matrices.) (15 points)

$$S: P_1 \rightarrow M_{1,2}, \quad S(a + bx) = [2a + 5b \quad a + 3b]$$



3. The function T below is a linear transformation (you may assume this). (45 points)

$$T: \mathbb{C}^3 \rightarrow \mathbb{C}^4, \quad T \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a + b - c \\ -5a - 4b + c \\ -2a - 2b + 3c \\ -a - c \end{bmatrix}$$

(a) Compute the preimage of $\begin{bmatrix} -1 \\ -1 \\ 3 \\ -3 \end{bmatrix}$, $T^{-1} \left(\begin{bmatrix} -1 \\ -1 \\ 3 \\ -3 \end{bmatrix} \right)$.

(b) Compute the kernel of T , $\mathcal{K}(T)$.

(c) Compute the range of T , $\mathcal{R}(T)$.

(d) Is T injective? Explain why.

(e) Is T surjective? Explain why.

(f) Is T invertible? Explain why.

(g) State a simple, but fundamental theorem about the rank and nullity of any linear transformation. Determine all of the relevant quantities for T , with justification, and then verify the conclusion of the theorem.



4. Illustrate the defining conditions of a linear transformation by proving that S below is a linear transformation. (P_1 is the vector space of polynomials with degree 1 or less.) (15 points)

$$S: \mathbb{C}^2 \rightarrow P_1, \quad T \left(\begin{bmatrix} a \\ b \end{bmatrix} \right) = (2a - b) + (3a + 2b)x$$

5. Suppose that $T: U \rightarrow V$ is a linear transformation. We will say that vectors $\mathbf{x}, \mathbf{y} \in U$ are “related” if $\mathbf{x} - \mathbf{y} \in \mathcal{K}(T)$. Notation for this relation is $\mathbf{x} \sim \mathbf{y}$. In other words, $\mathbf{x} \sim \mathbf{y}$ if and only if $\mathbf{x} - \mathbf{y} \in \mathcal{K}(T)$. Prove the three conditions that establish that T is an equivalence relation. (15 points)

(a) $\mathbf{x} \sim \mathbf{x}$ for all $\mathbf{x} \in U$.

(b) If $\mathbf{x} \sim \mathbf{y}$, then $\mathbf{y} \sim \mathbf{x}$.

(c) If $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} \sim \mathbf{z}$, then $\mathbf{x} \sim \mathbf{z}$.

