Math 290 Name: Key Dr. Beezer
Exam 5 Spring 2017
Chapters D&E

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
Partial credit is proportional to the quality of your explanation. Available Sage commands, or prohibitions, are present in each question. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Compute the determinant of B, $\det(B)$, without any use of Sage. (10 points)

$$B = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 3 \\ 2 & 2 & 5 \end{bmatrix}$$

Expand about Row 2 (Column 2 would be quick also.)

$$\det B = -(-1)\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} + 0\cdot(\;) - (3)\begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 1 - 3\cdot 4 = -11$$

2. Compute the eigenvalues of C, without any use of Sage. (15 points)

$$C = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$$

$$p_C(x) = \det(C - xI_2) = \begin{vmatrix} 1-x & 3 \\ 3 & 2-x \end{vmatrix}$$

$$= (1-x)(2-x) - 9 = 2 - 3x + x^2 - 9 = x^2 - 3x - 7$$

Eigenvalues are roots of $p_C(x) = 0$. Solve $x^2 - 3x - 7 = 0$ using the quadratic equation:

$$\lambda = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-7)}}{2} = \frac{1}{2}\left(3 \pm \sqrt{37}\right)$$

3. Answer the following questions about the matrix A. (45 points)

$$A = \begin{bmatrix} -10 & -3 & 27 & -3 & -39 & -24 \\ -30 & -37 & 99 & 87 & -3 & -60 \\ -12 & -9 & 35 & 15 & -21 & -24 \\ -3 & -9 & 15 & 26 & 15 & -6 \\ -3 & 0 & 6 & -3 & -10 & -6 \\ 0 & -3 & 3 & 9 & 9 & 2 \end{bmatrix}$$

(a) Use Sage to compute the eigenvalues of A. You may use any Sage commands you like.

A.eigenvalues() gives $[-1, -1, 2, 2, 2, 2]$

(b) Compute the algebraic and geometric multiplicity of each eigenavalue of A. You may use any Sage commands you like.

From (a): $\alpha_A(-1) = 2$, $\alpha_A(2) = 4$

Output of A.eigenmatrix_right() shows basis vectors to give
$\gamma_A(-1) = 2$, $\gamma_A(2) = 4$

(c) Compute a basis for the eigenspace of one of the eigenvalues, using Sage only to manipulate and row-reduce matrices.

$\lambda = -1$ will be less writing

$$(A - (-1)I_6).\text{rref}() = \begin{bmatrix} 1 & 0 & 0 & 0 & -4 & -1 \\ 0 & 1 & 0 & 0 & -10 & -13 \\ 0 & 0 & 1 & 0 & -4 & -3 \\ 0 & 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathcal{E}_A(-1) = \left\langle \left\{ \begin{bmatrix} 4 \\ 10 \\ 4 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 13 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

(d) Illustrate the use of a theorem (not Sage) to determine that A is diagonalizable.

$\alpha_A(-1) = 2 = \gamma_A(-1)$ So by Theorem DMFE, A is diagonalizable
$\alpha_A(2) = 4 = \gamma_A(2)$

(e) With no further use of Sage, use your computations above to provide a diagonal matrix that is similar to A.

The constructive proof of Theorem DC shows

$$D = \begin{bmatrix} -1 & & & & & 0 \\ & -1 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 2 & \\ 0 & & & & & 2 \end{bmatrix}$$

(f) With no further use of Sage, compute (by hand) the determinant of A.

$$\det(A) = \det(S^{-1}DS) = \det(S^{-1})\det(D)\det(S)$$
$$= \det(S^{-1})\det(S)\det D = \det(S^{-1}S)\det(D)$$
$$= \det(I_6)\det(D) = 1(-1)(-1)(2)(2)(2)(2) = 16$$

upper triangular (lower too!)

4. Similarity of square matrices is an equivalence relation. For this relation, prove the "transitive" part of the three-part definition of an equivalence relation (15 points)

Use $\sim$ for "similar to". We want to show:

$A \sim B$ & $B \sim C \Rightarrow A \sim C$.

$A \sim B \Rightarrow$ there exists S so that $B = S^{-1}AS$

$B \sim C \Rightarrow$ there exists T so that $C = T^{-1}BT$.

Then $C = T^{-1}BT = T^{-1}S^{-1}AST = (ST)^{-1}A(ST)$

sox & shoes

So A is similar to C via ST.

5. Suppose that A is a nonsingular matrix and λ is an eigenvalue of A. Prove that $1/\lambda$ is an eigenvalue of A^{-1}. (15 points)

Let $\underset{\sim}{x}$ be an eigenvector of A for λ. Then

$$A^{-1}\underset{\sim}{x} = \left(\tfrac{1}{\lambda}\lambda\right) A^{-1}\underset{\sim}{x}$$

if $\lambda \neq 0$, but this is never the case for a non singular matrix

$$= \tfrac{1}{\lambda} A^{-1}(\lambda \underset{\sim}{x})$$

$$= \tfrac{1}{\lambda} A^{-1} A\underset{\sim}{x}$$

$$= \tfrac{1}{\lambda} I \underset{\sim}{x}$$

$$= \tfrac{1}{\lambda} \underset{\sim}{x}$$

which demonstrates that $1/\lambda$ is an eigenvalue of A^{-1}.