Math 290 Name: Key Dr. Beezer
Exam 2 Spring 2017
Chapter V

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. You may use Sage to manipulate and row-reduce matrices. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features).

1. Is the vector $\mathbf{x}$ an element of the span of S, $\langle S\rangle$? Explain carefully why, or why not. (15 points)

$$\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} \qquad S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 1 \\ 1 \end{bmatrix} \right\} = \{v_1, v_2, v_3\}$$

① Scalars $\alpha_1, \alpha_2, \alpha_3$, so that $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = x$?

② SLSLC implies $A = [v_1 | v_2 | v_3]$, solution to $LS(A, x)$?

③ $$[A | x] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$, so ④ by RCLS, the system is inconsistent, and so $x \notin \langle S\rangle$

pivot column in last column

2. Write a nontrivial relation of linear dependence on T, or explain why no such thing exists. (15 points)

$$T = \left\{ \begin{bmatrix} -3 \\ -1 \\ 2 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -4 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 5 \\ -3 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -2 \\ -8 \\ 7 \end{bmatrix} \right\} = \{u_1, u_2, u_3, u_4\}$$

Check linear independence with Theorem LIVRN

$$A = [u_1 | u_2 | u_3 | u_4] \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r = 4 = n$, so T is linearly independent & there is only a trivial relation of linear dependence.

Answer:

3. Use the appropriate theorem to find a set P that is (1) a subset of S, (2) linearly independent, and (3) the span of P equals the span of S, $\langle P\rangle = \langle S\rangle$. (20 points)

$$S = \left\{ \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} -8 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} \right\}$$

Theorem BS suggests forming a matrix whose columns are the vectors of S.

$$= \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

$$B = [x_1 | x_2 | x_3 | x_4 | x_5 | x_6] \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -4 & 5 & 0 & -3 \\ 0 & 1 & 4 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Pivot columns $= D = \{1, 2, 5\}$

Answer:

$$P = \{x_1, x_2, x_5\}$$

$$= \left\{ \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 6 \end{bmatrix} \right\}$$

4. Given the matrix A, use the appropriate theorem to find a linearly independent set R so that the span of R is the null space of A, $\langle R\rangle = \mathcal{N}(A)$. (20 points)

$$A = \begin{bmatrix} -1 & 3 & -3 & 0 \\ 2 & -7 & 6 & -1 \\ -1 & 5 & -3 & 2 \\ 0 & -5 & 0 & -5 \\ -2 & 5 & -6 & -1 \end{bmatrix}$$

Row-reduce A to apply Theorem BNS.

$$\xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$D = \{1, 2\}$ $F = \{3, 4\}$

Solution vectors "look like"

$$x_3 \begin{bmatrix} \\ \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ 0 \\ 1 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Answer:

$$R = \left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

5. Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$. Give a proof that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ using a style that finishes with an application of Theorem CVE. (15 points)

For $1 \leq i \leq m$,

$$
\begin{aligned}
[\mathbf{u}+\mathbf{v}]_i &= [\mathbf{u}]_i + [\mathbf{v}]_i && \text{Defn CVA} \\
&= [\mathbf{v}]_i + [\mathbf{u}]_i && \text{Property CACN} \\
&= [\mathbf{v}+\mathbf{u}]_i && \text{Defn CVA}
\end{aligned}
$$

So, by Defn CVE, $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$.

6. Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a set of orthogonal vectors from $\mathbb{C}^m$. Prove that $\mathbf{v}_1$ is orthogonal to $2\mathbf{v}_2 + 5\mathbf{v}_3$. (15 points)

Check the inner product,

$$
\begin{aligned}
\langle \mathbf{v}_1, 2\mathbf{v}_2 + 5\mathbf{v}_3 \rangle &= \langle \mathbf{v}_1, 2\mathbf{v}_2 \rangle + \langle \mathbf{v}_1, 5\mathbf{v}_3 \rangle && \text{Theorem IPVA} \\
&= 2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle + 5\langle \mathbf{v}_1, \mathbf{v}_3 \rangle && \text{Theorem IPSM} \\
&= 2 \cdot 0 + 5 \cdot 0 && \text{Orthogonal set} \\
&= 0
\end{aligned}
$$

So by Definition OV, the two vectors are orthogonal.