Explorations of the Rubik's Cube Group

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Whoever came up with this...
Has to be the geekiest geek of all.

Grandmaster Flash

Abstract

Invented in 1974 by Ernö Rubik, a professor of architecture living in Budapest, Hungary, the Rubik's Cube is now one of the most popular toys in the world. For its relative simplicity, it has an incredible amount of mathematical complexity, which can be best appreciated through an understanding of the underlying group structure of the cube. This paper is intended to give a thorough comprehension of how the Rubik's Cube Group can be constructed with just an understanding of some group theory and the cube itself. In addition, it will explore some of the subgroups of the Rubik's Cube Group that are particularly relevant to solving the cube.

1 Introduction

When Ernö Rubik set out to build his cube, it was intended to be a an impossible structure. It could be twisted and rotated in every direction, but never fell apart. What he could not have known is that he had also created a toy that would one day go on to sell millions, and an object of considerable interest to mathematicians.

In fact, we partially have mathematicians to thank for the spread of the Rubik's Cube out of Hungary, as it was Hungarian mathematicians bringing their own cubes to international conferences that originally sparked the toy's appeal across nations. That interest has not waned in the last forty years, and even today new discoveries about the cube are being made, particularly now that computer simulations can run more and more quickly by the passing day. [**1**]

2 Vocabulary

Before we can dive into what sorts of structures give rise to the group itself, it's going to be important to lay out some of the basics of both group theory, and the notation and vocabulary used by "Cubers", those particularly devoted to the study or competitive solving of the Rubik's Cube. At this point, if you happen to own a Rubik's Cube, it will be helpful to have it on hand.

A Cube itself is made out of 27 small sections called "*cublets*". Each cubelet comes in one of three varieties, *corner, side*, and *center* cubelets.

A Cube itself has 6 faces, each broken up into 9 small sections called "*facelets*". Corner, edge, and center cubelets can be distinguished by their number of facelets. Corners have three, edges two, and centers just one. [**2**]

While a solved cube appears relatively simple, there are in fact $43252003274489856000 = 8! \times 3^7 \times (12!/2) \times 2^1 1$ possible unique states that a cube can be in.

As for the mechanics of the cube, a cursory observation reveals that each of the cube's six faces can rotate at 90° increments, allowing for an infinite number of rotations in any combination. We can denote these rotations as follows:

First begin by holding a cube in front of you. Without changing the

- F indicates a clockwise rotation of the side side facing you.
- B indicates a clockwise rotation of the side facing directly away from you.
- U indicates a clockwise rotation of the side facing straight up
- D indicates a clockwise rotation of the side facing straight down.
- R indicates a clockwise rotation of the side facing to your right.

- L indicates a clockwise rotation of the side facing to your left.

For any given rotation A, A^{-1} represents a rotation of the same face rotated by A in the counter-clockwise direction. For any series of rotations performed in sequence, they will be concatenated with the symbol '$*$'. For example, "rotate the top face, then right face" is notated as "U$*$R". Any rotation A repeated n times we write as "A^n". Lastly, let $1 = \mathrm{A}^0$ represent the identity rotation, which rotates the side by $0°$.

This is a slightly modified version of the *Singmaster Notation* for cubing developed by David Singmaster. By using a "$^{-1}$" instead of a prime symbol to indicate the inverse of a rotation, we can more easily make the transition into the group theory underlying the cube. [**2**]

With some experimentation, a few properties can be determined. First, for any rotation A, $\mathrm{A}*\mathrm{A}^{-1}$ is the same as having performed no action at all. Additionally, for any rotation A, A^3 is the same as A^{-1}. It's also clear that for any two adjacent faces, A$*$B is not the same as B$*$A. Lastly, no matter what kinds of rotations are made, the center facelets always remain in the same positions relative to each other.

3 Permutation Groups

At this point, it should be clear that there is some kind of group action happening on the cube, so let's briefly go over some of the things we'll need moving forward.

Recall that the permutations of a set form a group, and the following theorem is true:

Theorem 1. *The symmetric group on n letters, S_n, is a group with $n!$ elements, where the binary operation is the composition of maps.*

Now imagine a Rubik's Cube with each of the non-center facelets labeled 1 through 48. By performing rotations, we can change the positions of each of the numbered facelets. The discerning algebraist should be able to recognize this as a set, and a collection of permutations on that set.

With this in mind, let's redefine some of the rotations that we gave names to earlier. First, let S_{48} be the Symmetric group on 48 elements. Then, for the symbols R, L, D, F, U, and B, let

R = (3,38,43,19)(5,36,45,21)(8,33,48,24)(25,27,32,30)(26,29,31,28)
L = (1,17,41,40)(4,20,44,37)(6,22,46,35)(9,11,16,14)(10,13,15,12)
D = (14,22,30,38)(15,23,31,39)(16,24,32,40)(41,43,48,46)(42,45,47,44)
F = (6,25,43,16)(7,28,42,13)(8,30,41,11)(17,19,24,22)(18,21,23,20)

U = (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)
B = (1,14,48,27)(2,12,47,29)(3,9,46,32)(33,35,40,38)(34,37,39,36)
Using the cycle notation for permutations.

You are free to confirm for yourself that these permutations do in fact represent the six rotations of the cube, but I would recommend taking it at face value, for now. From this, it should be clear that the permutation group generated by our six side rotations, ⟨R, L, D, F, U, B⟩, is the Rubik's Cube Group.

While this does not tell us much on its own, it does provide for us a mathematical basis for talking about the Cube. A few simple conclusions can be made from here.

For example, (and this is somewhat obvious) the rotations of a single side are isomorphic to the cyclic group on 4 elements, $\mathbb{Z}_4$. To show this, suppose $\phi : \mathbb{Z}_4 \to G$, where G = ⟨ R ⟩ such that $\phi(x) =$

$$\begin{cases} () & x = 0 \\ R^x & x \neq 0 \end{cases}$$

With a small amount of computation, or the aid of a convenient mathematics software, it can be easily shown that ϕ is an isomorphism. Similarly, this can be shown for the other five generators. [**3**]

This is just one of countless subgroups of the Rubik's Cube Group that we could explore. In fact, every element you can create through a series of possible rotations generates a subgroup. Considering that the number of possible permutations of the Cube is upwards of forty-three quintillion, it is not feasible to explore all of them here. However, we can study some small, simple subgroups in closer detail.

4 The Slice Group

Suppose that with a cube held in front of you, you rotate both the right and left sides up, or perform the permutation R$*$L^{-1}(=L^{-1}R). It might be easier to instead imagine rotating the center "slice" of the cube down by a quarter turn. Consider the subgroup generated by this rotation, along with the rotations U$*$D^{-1} and F$*$B^{-1} This subgroup, called the Slice Group, is of significance to Cubers because it can be used to easily create interesting and symmetric flower like patterns, but can also be explored by us in some amount of depth. For example, let's see if we can determine a more detailed look at the structure of H.

First, let's recall a definition that will be important for doing this.

Definition 4.1. Group Action An action of a group G on a set X is a map $G \times X \to X$ given by $(g, x) = gx$ where $ex = x$ for all x where e is the identity element of the group, and $(g_1 g_2)x = g_1(g_2 x)$ for all $x \in X$ and all $g_1, g_2 \in G$

Let $M_R = R * L^{-1}, M_U = U * D^{-1}$ and $M_F = F * B^{-1}$, and let H be the group $\langle M_R, M_U, M_F \rangle$.

Now let E be the set of edge cubelets, and C be the set of center facelets, and let $X = E \cup C$. Notice that X is a set, not a group. H, then, acts on the set X. Observe that H does not alter any of the corner elements of the cube set. You can confirm this with a cube of your own by observing that no move in the slice group changes the relative positions of the corners to each other. [**3**]

Now suppose that we call two edge cubelets equivalent if one can be sent to the other via an element of H. Then H partitions X into three orbits, the edges along the RL slice, UD slice, and FB slice, denoted E_{RL}, E_{UD} and E_{FB}, respectively.

Since H acts on the set E_{RL}, we have a homomorphism $r_{RL} : H \to S_{E_{RL}}$, where $S_{E_{RL}}$ is the symmetric group on E_{RL}, which takes each $M_A \in H$ to M_R. Similarly, we have $r_{FB} : H \to S_{E_{FB}}$ and $r_{UD} : H \to S_{E_{UD}}$. H also acts on each of the sets E and C, which gives us the homomorphisms

$$r = r_{RL} \times r_{FB} \times r_{UD} : H \to S_{E_{RL}} \times S_{E_{FB}} \times S_{E_{UD}} \subset S_E$$

and $s : H \to S_C$, which we can combine into a single homomorphism

$$r \times s : H \to S_{E_{RL}} \times S_{E_{FB}} \times S_{E_{UD}} \times S_C$$

Next consider that the image of H in $S_{E_{RL}}$ is $\langle M_R \rangle$, with similar results for $S_{E_{FB}}$, and $S_{E_{UD}}$. Recalling how we proved that a single rotation of the side of a cube is isomorphic to $\mathbb{Z}_4$, the cyclic group on 4 elements, it is simple to show that each of these groups is isomorphic to $\mathbb{Z}_4$ as well.

Next we'll take a look at the image of H in S_C. Looking at the entire cube from the perspective of just the center facelets, this becomes surprisingly simple. Each movement, as far as the centers are concerned, is no different from a rigid rotation of the entire cube. Thus, the image of H in S_C is simply the rotation group of a cube. This, conveniently provided to us by David Joyner, is simply S_4.

Combining all this information, we can see that the image of H in $S_{E_{RL}} \times S_{E_{FB}} \times S_{E_{UD}} \times S_C$ is isomorphic to a subgroup of

$$C_4^3 \times S_4$$

A considerably more detailed and interesting result than simply recognizing it as a subgroup of S_{48}

5 The Illegal Rubik's Cube Group

Next we'll take a step up and out of the Rubik's Cube group and take a look at a group that contains it as a subgroup.

Before we can do that, however, we need to define two new terms. [**3**]

Definition 5.1. Semi-Direct Product

Suppose that H_1 and H_2 are both subgroups of a group G. We say that G is the semi-direct product of H_1 by H_2, written $H_1 \rtimes H2$ if

- $G = H_1 \times H_2$
- H_1 and H_2 only have the identity of G in common
- H_1 is normal in G

Definition 5.2. Wreath Product

Let G_1 be a group, and let G_2 be a group acting on a finite set X, where $|X| = m$, and let G_1^X denote the direct product of G_1 with itself m times, with the coordinates labeled by the elements of X. The wreath product of two groups G_1 and G_1 is the group G_1 wr $G_2 = G_1^X$ where the action of G_2 on G_1^X is via its action on X.

Let H be the Illegal Rubik's Cube Group, which is differentiated from the Rubik's Cube Group by the fact that it allows for "disassembling" and "reassembling" the cubelets of the cube. By doing this, it is possible to produce an larger group that contains the Rubik's Cube group as a subgroup.

Now consider any given corner cubelet. Now that we are free from the constraint of whether a given permutation is "legal", we can describe all of its possible positions more simply. Because we are not removing the stickers from the cubelet, the relative positions of the three facelets do not change. However, it can rotate freely. Since there are three facelets, this forms the cyclic group on 3 elements, $\mathbb{Z}_3$. Because there are eight possible corners of the cube a corner cubelet can occupy, the orientation of a single facelet of a corner cubelet in the Illegal Rubik's Cube group can be described by the direct product of $\mathbb{Z}_3$ with itself eight times, $\mathbb{Z}_4^8$. Because there are eight corner cubes that can be arranged over eight positions, the possible arrangements can be described by the permutation group on eight elements, S_8.

The wreath product of these groups, $\mathbb{Z}_3^8$ wr S_8, describes the position of all corner facelets in the Illegal Cube group.

Similarly, we can determine that for an edge cubelet, there are 12 possible locations on the cube that a given edge cubelet can occupy, and 2 orientations that it can have. Therefore, the group $\mathbb{Z}_2^{12}$ describes all the possible orientations of a single edge cubelet.

Again, because there are 12 edge facelets and 12 possible positions they can hold, their arrangements can be described by S_{12}, and again, the wreath product of these groups $\mathbb{Z}_2^{12}$ wr S_{12} describes the positions of all edge facelets in the Illegal Cube group.

The direct product of these two groups, $(\mathbb{Z}_3^8 \text{ wr } S_8) \times (\mathbb{Z}_2^{12} \text{ wr } S_{12})$ describes the entirety of the Illegal Rubik's Cube group, a result that follows from the previous group constructions.

6 Sources Cited

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