The Hamiltonian quaternions $\mathbb{H}$ are a system of numbers devised by William Hamilton in 1843 to describe three dimensional rotations.

- $q = a + bi + cj + dk$ where $i^2 = j^2 = k^2 = ijk = -1$
- non-abelian multiplication
Conjugation and Norms

- Conjugation in the Hamiltonian quaternions is defined as follows: if \( q = a + bi + cj + dk \) then \( \overline{q} = a - bi - cj - dk \).
- The norm is defined by
  \[
  N(q) = q\overline{q} = \overline{q}q = a^2 + b^2 + c^2 + d^2.
  \]
Some important properties of the conjugate and norm.

- $\overline{\bar{q}} = q$
- $\overline{q_1 + q_2} = \overline{q_1} + \overline{q_2}$
- $\overline{q_1 q_2} = \overline{q_2} \hspace{1mm} \overline{q_1}$

Elements with nonzero norms have multiplicative inverses of the form $\frac{q}{N(q)}$.

The norm preserves multiplication

$$N(q_1 q_2) = q_1 q_2 \overline{q_1 q_2} = q_1 q_2 \overline{q_2} \overline{q_1} = q_1 N(q_2) \overline{q_1}$$

$$= N(q_2)q_1 \overline{q_1} = N(q_2)N(q_1)$$
Definition of an Algebra

An algebra over a field is a vector space over that field together with a notion of vector multiplication.
Generalizing the Quaternions

The Hamiltonian quaternions become a prototype for the more general class of quaternion algebras over fields. Defined as follows:

- A quaternion algebra \((a, b)_F\) with \(a, b \in F\) is defined by \(\{x_0 + x_1i + x_2j + x_3k | i^2 = a, j^2 = b, ij = k = -ji, x_i \in F\}\).

- Under this definition we can see that \(\mathbb{H} = (-1, -1)_\mathbb{R}\) since

\[
k^2 = (ij)^2 = ijij = -iijj = -(-1)(-1) = -1
\]

- Note: We will always assume that \(char(F) \neq 2\).
Conjugation works the same $\bar{q} = x_0 - x_1i - x_2j - x_3k$

The Norm is defined as
$N(q) = \bar{q}q = q\bar{q} = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$, it still preserves multiplication.

Inverse elements are still defined as $\frac{\bar{q}}{N(q)}$ for elements with nonzero norms.
The split-quaternions are the quaternion algebra \((1, -1)_{\mathbb{R}}\).

- Allows for zero divisors and nonzero elements with zero norms

\[(1 + i)(1 - i) = 1 + i - i - 1 = 0\]
An isomorphism between quaternion algebras is a ring isomorphism that fixes the ”scalar term”.

For example:

\[
1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, i \rightarrow \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, j \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, k \rightarrow \begin{bmatrix} 0 & -1 \\ a & 0 \end{bmatrix}
\]

is an isomorphism from any quaternion algebra \((a, 1)_{\mathbb{F}}\) to \(M_2(\mathbb{F})\) the algebra of 2 × 2 matrices over \(\mathbb{F}\).
A quaternionic basis is a set \( \{1, e_1, e_2, e_1 e_2\} \) where \( e_1^2 \in F \), \( e_2^2 \in F \), \( e_1^2, e_2^2 \neq 0 \), and \( e_1 e_2 = -e_2 e_1 \).

Isomorphisms between quaternion algebras can be determined through the construction of quaternionic bases. If you can construct bases in two algebras such that the values of \( e_1^2 \) and \( e_2^2 \) are equal, then those algebras are isomorphic to one another.

- This shows that \((a, b)_F\), \((b, a)_F\), \((a, -ab)_F\) and all similar permutations of \( a, b \), and \(-ab\) produce isomorphic algebras.
Important Categories of Isomorphism

- \((a, b^2)_F \cong M_2(F)\)
- Since an isomorphism exists:

\[
1 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i \rightarrow \begin{bmatrix} 0 & 1 \\ a & 0 \end{bmatrix}, \quad j \rightarrow \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix}, \quad k \rightarrow \begin{bmatrix} 0 & -b \\ ab & 0 \end{bmatrix}
\]
(a, b)_F \cong M_2(F) \text{ if } b = x^2 - ay^2 \text{ for } x, y \in F

To show this we construct a basis \{1, i, jx + ky, (i)(jx + ky)\}, this is clearly a basis of \((a, b)_F\) and since

\[(jx + ky)^2 = j^2x^2 + jkxy + kjxy + k^2y^2\]

\[= bx^2 - aby^2 = b(x^2 - ay^2) = b^2\]

It is also a basis of \((a, b^2)_F\) so

\[(a, x^2 - ay^2)_F \cong (a, b^2)_F \cong M_2(F).\]
The Norm Subgroup

Elements of a field of the form $x^2 - ay^2$ for a given $a$ form a group under multiplication known as the norm subgroup associated to $a$ or $N_a$.

- $1 = 1^2 - a0^2$
- $(x^2 - ay^2)(w^2 - az^2) = (xw + ayz)^2 - a(xz + wy)^2$

$$\frac{1}{x^2 + ay^2} = \frac{x^2 + ay^2}{(x^2 + ay^2)^2} = \frac{x}{x^2 + ay^2}^2 - a\frac{y}{x^2 + ay^2}^2$$
Real Quaternion Algebras

**Theorem:** There are only two distinct quaternion algebras over $\mathbb{R}$ which are $\mathbb{H}$ and $M_2(\mathbb{R})$.

**Proof:**

- Given $(a, b)_\mathbb{R}$ if $a, b < 0$ then we can construct a basis $\{1, \sqrt{-ai}, \sqrt{-bj}, \sqrt{abij}\}$ in $\mathbb{H}$ which forms a basis of $(a, b)_\mathbb{R}$ indicating the existence of an isomorphism.

- If $a > 0, b < 0$ WLOG, we can construct a basis $\{1, \sqrt{ai}, \sqrt{-bj}, \sqrt{-abij}\}$ in the $(1, -1)_\mathbb{R}$ which forms a basis of $(a, b)_F$ indicating the existence of an isomorphism with the split-quaternions and therefore $M_2(F)$. 
Complex Quaternion Algebras

**Theorem:** There is only one quaternion algebra over \( \mathbb{C} \), which is \( M_2(\mathbb{C}) \).

**Proof:**

- We’ve shown that \( (a, b^2)_F \cong M_2(F) \). We can find always find a \( c \in \mathbb{C} \) such that \( c^2 = b \), therefore \( (a, b)_\mathbb{C} \cong (a, c^2)_\mathbb{C} \cong M_2(\mathbb{C}) \).
Theorem: All quaternion algebras that are not division rings are isomorphic to $M_2(F)$.

Proof: Take a quaternion algebra $A = (a, b)_F$

- If $a = c^2$ or $b = c^2$ for some $c \in F$ then $A \cong M_2(F)$, now assume neither $a$ nor $b$ are squares.

- If $A$ isn’t a division ring then there must be some nonzero element without a multiplicative inverse. We will show that $b = x^2 - ay^2$ and therefore $(a, b)_F \cong M_2(F)$. 
The only elements without inverses are those with
\[ N(q) = x_1^2 - ax_2^2 - bx_3^2 + abx_4^2 = 0 \]
\[ x_1^2 - ax_2^2 = b(x_3^2 - ax_4^2) \]
\[ x_3^2 - ax_4^2 \neq 0 \] since either \( x_3 = x_4 = 0 \) or \( a = \frac{x_3^2}{x_4^2} \). If \( x_3 = x_4 = 0 \) then either \( x_1 = x_2 = 0 \) or \( a = \frac{x_1^2}{x_2^2} \). All of which are contradictions.
So \( b = \frac{x_1^2 - ax_2^2}{x_3^2 - ax_4^2} \), therefore \( b = x^2 - ay^2 \) by closure of \( N_a \) so \( A \cong M_2(F) \).
Rational Quaternion Algebras

It can be shown that there are infinite distinct quaternion algebras over $\mathbb{Q}$. By the previous theorem all but $M_2(\mathbb{Q})$ must be division rings.
The octonions are another set of numbers, discovered independently by John T. Graves and Arthur Cayley in 1843, which are of the form:

\[ o = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 \]

- Multiplication neither commutative nor associative
- Obeys the Moufang Identity \((z(x(zy))) = (((zx)z)y)\), weaker than associativity but behaves similarly.
- Conjugation behaves the same.
- Norm still preserves multiplication.
Figure: The Fano plane
Much as quaternion algebras can be described by \((a, b)_F\)
octonion algebras can be described by three of their seven in the form \((a, b, c)_F\).

- \((-1, -1, -1)_\mathbb{R}\) are Graves’ octonions
- \((1, 1, 1)_\mathbb{R}\) are the split-octonions
- these are the only two octonion algebras over \(\mathbb{R}\)
Unlike the quaternions, octonions and by extension octonion algebras cannot be expressed as matrices since matrix multiplication is associative. German mathematician Max August Zorn created a system called a vector-matrix algebra which could be used to describe them.

\[
\begin{bmatrix}
    a & u \\
    v & b
\end{bmatrix}
\begin{bmatrix}
    c & w \\
    x & d
\end{bmatrix} =
\begin{bmatrix}
    ac + u \cdot x & aw + du - v \times x \\
    cv + bx + u \times w & bd + v \cdot w
\end{bmatrix}
\]
Two complex elements that are not scalar multiples of one-another generate a quaternion subalgebra.

Information about isomorphisms is less readily available, it’s clear that some of the same principles apply but with added difficulty.

Sedenion algebras (16-dimensional) and above cease being composition algebras.