Math 290 Name: Key Dr. Beezer
Exam 7 Spring 2016
Chapter R

Show *all* of your work and *explain* your answers fully. There is a total of 100 possible points.
For computational problems, place your answer in the provided boxes. Partial credit is proportional to the quality of your explanation. Unless the problem directions specify otherwise, you may use Sage to manipulate matrices and vectors, row-reduce matrices, compute determinants, compute and factor characteristic polynomials, and compute eigenvalues and eigenspaces. No other use of Sage may be used as justification for your answers. When you use Sage be sure to explain your input and show any relevant output (rather than just describing salient features of something I can't see).

1. Find the matrix representation of S relative to the provided bases of the vector space P_1 (polynomials of degree at most 1) and $\mathbb{C}^3$ (column vectors with three entries). (15 points)

$$S\colon P_1 \to \mathbb{C}^3,\ S\left(a+bx\right) = \begin{bmatrix} 2a+b \\ -a+4b \\ 3a-2b \end{bmatrix} \qquad B = \{3+x,\ 7+2x\} \qquad C = \left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

$$\rho_C(S(3+x)) = \rho_C\left(\begin{bmatrix} 7 \\ 1 \\ 7 \end{bmatrix}\right) = \rho_C\left(6\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 6\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}$$

$$\rho_C(S(7+2x)) = \rho_C\left(\begin{bmatrix} 16 \\ 1 \\ 17 \end{bmatrix}\right) = \rho_C\left(18\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 19\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 18 \\ -1 \\ 19 \end{bmatrix}$$

Answer:

$$M^S_{B,C} = \begin{bmatrix} 6 & 18 \\ 1 & -1 \\ 6 & 19 \end{bmatrix}$$

2. Find a basis of P_2 composed entirely of eigenvectors of the linear transformation T. (20 points)

$T\colon P_2 \to P_2, \qquad T\left(a+bx+cx^2\right) = (-3a+5b+10c) + (10a-8b-20c)\,x + (-5a+5b+12c)\,x^2$

$B = \{1, x, x^2\}$, "nice" basis, use for a matrix representation

$$M^T_{B,B} = \begin{bmatrix} -3 & 5 & 10 \\ 10 & -8 & -20 \\ -5 & 5 & 12 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$ Apply ρ_B^{-1}, uncoordinatize,

use .eigenmatrix_right()

$\lambda = -3, 2, 2$; scaled eigenvector, linearly independent set (so a basis of $\mathbb{C}^3$)

Answer:

$$\hat{B} = \{1-2x+x^2,\ 2+x^2,\ 2x-x^2\}$$

3. Consider the linear transformation T below, between vector spaces of column vectors, $\mathbb{C}^3$ and $\mathbb{C}^2$, along with two bases for each vector space. (35 points)

$$B = \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix} \right\} \quad C = \left\{ \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}7\\3\\1\end{bmatrix}, \begin{bmatrix}4\\1\\-4\end{bmatrix} \right\} \quad D = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\} \quad E = \left\{ \begin{bmatrix}1\\-2\end{bmatrix}, \begin{bmatrix}2\\-3\end{bmatrix} \right\}$$

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \qquad T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}3a+b\\a-2b\end{bmatrix}$$

(a) Compute the matrix representation of T relative to B and D.

$$T(\underset{\sim}{b_1}) = \begin{bmatrix}3\\1\end{bmatrix} \xrightarrow{\rho_D} \begin{bmatrix}3\\1\end{bmatrix}$$
$$T(\underset{\sim}{b_2}) = \begin{bmatrix}4\\-1\end{bmatrix} \xrightarrow{\rho_D} \begin{bmatrix}4\\-1\end{bmatrix}$$
$$T(\underset{\sim}{b_3}) = \begin{bmatrix}4\\-1\end{bmatrix} \xrightarrow{\rho_D} \begin{bmatrix}4\\-1\end{bmatrix}$$

Answer:
$$M^T_{B,D} = \begin{bmatrix}3 & 4 & 4\\1 & -1 & -1\end{bmatrix}$$

(b) Compute the matrix representation of T relative to C and E directly (i.e. without duplicating computations below).

$$T(\underset{\sim}{c_1}) = \begin{bmatrix}7\\0\end{bmatrix} \xrightarrow{\rho_E} \begin{bmatrix}-21\\14\end{bmatrix}$$
$$T(\underset{\sim}{c_2}) = \begin{bmatrix}24\\1\end{bmatrix} \longrightarrow \begin{bmatrix}-74\\49\end{bmatrix}$$
$$T(\underset{\sim}{c_3}) = \begin{bmatrix}13\\2\end{bmatrix} \longrightarrow \begin{bmatrix}-43\\28\end{bmatrix}$$

Answer:
$$M^T_{C,E} = \begin{bmatrix}-21 & -74 & -43\\14 & 49 & 28\end{bmatrix}$$

(c) Compute the change-of-basis matrices $C_{B,C}$ and $C_{D,E}$.

Easier to do opposite & invert

$$C_{C,B} = \begin{bmatrix}1 & 4 & 3\\0 & 2 & 5\\1 & 1 & -4\end{bmatrix}$$ can almost do vectors in C as linear combos of B visually

$$C_{E,D} = \begin{bmatrix}1 & 2\\-2 & -3\end{bmatrix}$$ on sight

Answer:
$$C_{B,C} = \begin{bmatrix}-13 & 19 & 14\\5 & -7 & -5\\-2 & 3 & 2\end{bmatrix}$$
$$C_{D,E} = \begin{bmatrix}-3 & -2\\2 & 1\end{bmatrix}$$

(d) Perform a single computation that illustrates the relationship between *all* of the answers above and verifies their correctness.

One version, derive (b) from (a) & (c)

$$M^T_{C,E} = C_{D,E}\, M^T_{B,D}\, C_{C,B}$$
$$= \begin{bmatrix}-3 & -2\\2 & 1\end{bmatrix}\begin{bmatrix}3 & 4 & 4\\1 & -1 & -1\end{bmatrix}\begin{bmatrix}1 & 4 & 3\\0 & 2 & 5\\1 & 1 & -4\end{bmatrix} = \begin{bmatrix}-21 & -74 & -43\\14 & 49 & 28\end{bmatrix}$$

4. Suppose that V is a vector space with a bases B and C. Give a proof that the change-of-basis matrices, $C_{B,C}$ and $C_{C,B}$ are related according to $C_{C,B} = (C_{B,C})^{-1}$. Your proof should not just be quoting a theorem, but instead should involve the definition of a change-of-basis matrix and use other, more basic, theorems from this chapter or previous chapters. (15 points)

$$C_{C,B} = M^{I_V}_{C,B}$$ ← identity linear transformation on V

Definition

$$= \left(M^{I_V^{-1}}_{B,C}\right)^{-1}$$ Theorem IMR

$$= \left(M^{I_V}_{B,C}\right)^{-1}$$ Inverse of $I_V: V \to V$ is itself

$$= C_{B,C}^{-1}$$

5. Suppose that V is a vector space with a basis B. Prove that the linear transformation ρ_B is injective. (15 points)

$\rho_B : V \to \mathbb{C}^n$ $\quad n = \dim(V)$, write $B = \{u_1, u_2, \ldots, u_n\}$

$$\mathcal{K}(\rho_B) = \{v \mid \rho_B(v) = 0\}$$ (general)

$$= \left\{a_1 u_1 + a_2 u_2 \cdots + a_n u_n \,\middle|\, \begin{bmatrix} a_1 \\ a_2 \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$$ (specific)

$$= \{0u_1 + 0u_2 + \cdots + 0u_n\}$$ ← a set w/ one element

$$= \{0\}$$

By Theorem KILT, ρ_B is injective.